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Pure spinor superstrings on generic type IIA supergravity backgrounds*

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Abstract: We derive the Free Differential Algebra for type IIA supergravity in 10 dimensions in the string frame. We provide all fermionic terms for all curvatures. We derive the Green-Schwarz sigma model for type IIA superstring based on the FDA construction and we check its invariance under κ-symmetry. Finally, we derive the pure spinor sigma model and we check the BRST invariance. The present derivation has the advantage that the resulting sigma model is constructed in terms of the superfields appearing in the FDA and therefore one can directly relate a supergravity background with the corresponding sigma model. The complete explicit form of the BRST transformations is given and some new pure spinor constraints are obtained. Finally, the explicit form of the action is given.

Keywords: Superstrings and Heterotic Strings, BRST Symmetry, BRST Quantization.

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Contents

1. Introduction

The pure spinor formulation of superstrings is a new formalism [1] which powerfully uses the advantages of the RNS formulation and those of the GS formalism. In particular, the purpose of its creation was to provide a set-up where the RR fields (appearing in the spectrum of superstrings) could be treated on the same footing as the NSNS ones. This equal-footing treatment of the bosonic massless modes of superstrings is realized in every formulation of supergravity (in components, in superspace or, using rheonomic approach).

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Therefore it would be convenient also for the pure spinor sigma model. This means that the couplings of the worldsheet fields with the RR backgrounds must be very similar to the coupling with the NSNS fields. This is indeed achieved in the pure spinor formulation.

Dealing with the complete supergravity multiplet and with its non-linear self-interactions requires a full-fledged formulation of pure spinor superstrings on arbitrary (on-shell) background. This has been achieved in the fundamental work [2] where a generic sigma model, respecting the requirements of super-Poincaré invariance (both on the world-sheet as well as in the target space) and with the correct quantum numbers has been constructed. Consequently, according to the formulation, two BRST currents and their charges are provided. Thus, imposing the nilpotency of these BRST charges (which is equivalent to the closure of the constraint algebra) and the holomorphicity of their currents (which is equivalent to the invariance of the action), the authors derived the supergravity equations of motion in the form of superspace constraints. The main input in [2] is the requirement of the constraints on the ghost fields

$$\bar{\lambda}_1 \Gamma^m \lambda_1 = 0 \,, \qquad \bar{\lambda}_2 \Gamma^m \lambda_2 = 0 \,.$$

Here $\bar{\lambda}_i = \lambda_i^T C$ with C is the charge conjugation matrix. The index i stands for the right- or the left-mover pure spinors whose chirality is decided by choosing either IIA or IIB. These constraints are necessary for the nilpotency of the BRST charge in the flat limit and they are essential to establish the correct number of degrees of freedom. Therefore, they have been imposed also for the interacting sigma model on generic backgrounds. Doing that, the emerging superspace constraints have a complicated and unconventional relation with the standard description of supergravity. Yet, in [2] it is argued how, using Weyl superspace [3], one can relate the supergravity constraints from the pure spinor formulation with those given in [4, 5]. To be more explicit, the connection between a more conventional setting and the pure spinor formulation is obtained by a Weyl transformation involving the dilatino followed by a Poincaré transformation needed to reabsorb some additional terms in the variation of the gravitino fields. Thus, the conclusion is that, insisting on very simple pure spinor constraints, the ensuing supergravity parametrization in superspace turns out to be rather obscure. This fails to provide a practical and an effective algorithm to deduce the pure spinor sigma model starting from a given supergravity background.

Let us invert the path. The old path goes from pure spinor constraints to the sigma model and yields the supergravity constraints as a by-product. The new path goes from the geometrical formulation of supergravity to the pure spinor sigma model. Indeed, we decide to start from a convenient description of supergravity and deduce the constraints and the conditions under which a pure spinor sigma model can exist.

For these reasons it is highly desirable to have a formulation of the pure spinor sigma models in which the pure spinor constraints, the BRST operator and the entire set up follow from background supergravity as it happens for the κ-symmetric actions.

Such a formulation is presented in this paper. Previous work in this direction was accomplished in [6 − 8], where such ideas were applied to the case of M-theory and of the M2-brane. Here we focus on type II superstrings and in particular on the type IIA case.

This is not a random choice but it is motivated by precise reasons. Our ultimate goal is three-fold, since we want to show that:

1. The pure string BRST invariant σ-model can be constructed on arbitrary supergravity backgrounds;

2. The structure of the BRST algebra, the form of the pure spinor constraints and the 2-dimensional action can be algorithmically derived from supergravity and its Free Differential Algebra;

3. The explicit form of the σ-model action obtained in this way is of immediate practical use for dealing with backgrounds characterized by less than maximal supersymmetry, like AdS $\times$ M supergravity solutions where M is not a sphere.

As we already discussed in [7], issue 3) consists of solving the supergravity problem of *supergauge completion*. This means the explicit integration in superspace of the rheonomic conditions which are first order differential equations in the Grassmann θ-variables. Such integration is just a brute-force matter (see for example the application to super-Yang-Mills in 10d [9]), being a priori guaranteed by the fulfillment of Bianchi identities and, it can be quite cumbersome in general situations. In the directions of those θ-variables that correspond to supersymmetries preserved by the chosen background, the integration is automatically performed by the use of Maurer-Cartan superforms of the superisometry algebra (for instance $\mathrm{SU}(2,2|1)$ in the case of the $\mathrm{AdS}_5 \times \mathrm{T}^{(1,1)}$ compactification of type IIB supergravity [10 − 12] or $\mathrm{Osp}(6|4)$ in the case of the $\mathrm{AdS}_4 \times \mathbb{P}^3$ compactification of the type IIA theory [13]). In the other directions, namely those along the θ's associated with broken supersymmetries, the integration of the rheonomic conditions might be involved. Hence, in order to explore the structure of the *supergauge completion* it is desirable to have the minimal possible amount of broken thetas. Among the possible compactifications, one case is the $\mathrm{AdS}_4 \times \mathbb{P}^3$ background. There the preserved thetas are 24 and the broken ones just 8, and they are arranged into an $\mathrm{O}(2)$ doublet of $\mathrm{D} = 4$ spinors leading to the hope that the corresponding sigma model as a nice and insightful description. It is therefore in such perspective we began to focus on the type IIA case rather than on the type IIB one which will follow [14].

A second technical reason for this strategy will be clear to the reader. In order to carry through our programme, the formulation of supergravity which is required is in the *string frame* rather than that in the *Einstein frame*. Although the two formulations are simply related by a field redefinition, the implementation of such a change of variables in the rheonomic solution of the Free Differential Algebra Bianchi Identities is so cumbersome that it turns out to be more convenient to redo the construction of supergravity directly in the new frame. In view of this we can say that neither the type IIA nor the type IIB theory were available in the rheonomic framework and in the string frame when we started the present work. Indeed the rheonomic type IIA theory was never constructed, while the rheonomic type IIB case was constructed by Castellani and Pesando in the *Einstein frame* [15, 16]. The transition to the string frame is even more elaborate in the IIB case

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than in the IIA one, due to the SU(1, 1) covariance of the IIB theory, which is made manifest only in the Einstein frame.

Having clarified our motivations, let us summarize the structure of the paper:

1. As already recalled above, the algebraic structure underlying any higher dimensional supergravity theory is a Free Differential Algebra (FDA) [17, 18]. This latter is a categorical extension of a (super) Lie algebra determined by the Chevalley cohomology of the latter [19];

2. Given the FDA one considers its Bianchi identities and constructs the unique rheonomic parametrization of the FDA curvatures. Rheonomy is a universal principle of analiticity in superspace [20] which requires that the fermionic components of the FDA curvatures should be linear functions of their bosonic ones. Rheonomy encodes in one single principle the construction of both field equations and supersymmetry transformation rules for any supergravity. Indeed field equations follow as integrability conditions of the rheonomic parametrization of curvatures. The flow chart for the construction of classical supergravities was for instance recently presented in [8];

3. Consider then the FDA appropriate to the supergravity under investigation and the rheonomic parametrization of its curvatures;

4. Perform the ghost-form extension of the classical FDA according to the principle introduced by Anselmi and Fré in [21], which generalizes ideas previously introduced by Baulieu [22] namely:

   The BRST algebra is provided by replacing, in the rheonomic parametrization of the classical supergravity curvatures, each differential form with its extended ghost-form counterpart while keeping the curvature components untouched. Thus one obtains the rheonomic parametrization of the ghost-extended curvatures, whose formal definition is identical with that of the classical curvatures with the replacements:

   $$\begin{array}{rcl} d & \mapsto & d+\mathcal{S} \\ \Omega^{[n]} & \mapsto & \sum_{p=0}^{n} \Omega^{[n-p,p]} \end{array} \tag{1.1}$$

   where $\mathcal{S}$ is the BRST differential and $\Omega^{[n-p,p]}$ is a ghost form with form degree $n-p$ and ghost number p.

   In this way one has the ordinary (unconstrained) BRST algebra of supergravity;

5. Set to zero all the bosonic ghosts. This defines a constrained BRST algebra and for consistency a certain set of *pure spinor* constraints. The correct constraints are the projection onto the world-sheet (brane world volume) of these constraints.;

6. Verify that the pure spinor constraints can be solved in terms of as many independent degrees of freedom as it is required for a conformal theory in d=2 in the case of superstrings with vanishing central charge;

superPoincaré algebra

$\Downarrow$

FDA

$\Downarrow$

Rheonomic solution of FDA Bianchis

$\Downarrow$

BRST ghost-extension

$\Downarrow$

Restriction to fermionic ghosts

$\Downarrow$

Berkovits algebra and pure spinor constraints

Table 1: Derivation of the Berkovits algebra.

7. Introduce the appropriate antighosts and Lagrange multiplier field and construct the BRST invariant quantum action.

The whole procedure can be summarized in table 1.

In this way we determine a path from the superPoincaré algebra to the Berkovits BRST algebra on the fields of non negative ghost-number (see the above flowchart). As we pointed out in [8] the inclusion of the extra fields with negative ghost number (the antighosts) requires more explanation since it is not a standard gauge-fixing procedure but it is obviously essential for the construction of the σ-model action.

We explicitly show how to realize the last steps of the construction in the case of the type IIA theory and we emphasize that they are just possible because of some very special features of the rheonomic solution of the FDA Bianchi identities which are displayed by its *string frame* formulation and are instead absent in the *Einstein frame*.

The result of our construction is an explicit expression of the pure spinor BRST invariant action of type IIA superstrings holding true on any supergravity background, irrespectively of the number of supersymmetries it preserves. As a by-product of the construction we have also the emission vertices for all the supergravity fields, both fermionic and bosonic, both of the Neveu-Schwarz and of the Ramond Ramond sectors.

Our paper is organized as follows: In section 2, we discuss the formulation of supergravity using the Free Differential Algebra in the string frame. We compute the complete parametrization of the fermionic and bosonic curvatures, including the 3 and 4-fermion terms. In section 3, we construct the Green-Schwarz sigma model for type IIA superstring using the FDA and we discuss the background independence. In section 4, we provide the pure spinor formulation of superstring based on the BRST transformations obtained from the FDA. In appendices we supplement the main text with some detail of the derivation and the conventions.

2. Type IIA supergravity and its FDA

Free Differential Algebras (FDA) are a natural categorical extension of the notion of Lie

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algebra and constitute the natural mathematical environment for the description of the algebraic structure of higher dimensional supergravity theory, hence also of string theory. The reason is the ubiquitous presence in the spectrum of string/supergravity theory of antisymmetric gauge fields (p-forms) of rank greater than one.

FDA.s were independently discovered in Mathematics by Sullivan [19] and in Physics by two of the authors of this paper (R. D'Auria and P. Fré) [17]. The original name given to this algebraic structure by D'Auria and Fré was that of *Cartan Integrable Systems*. Later, recognizing the conceptual identity of this supersymmetric construction with the pure bosonic constructions considered by Sullivan, we also turned to its naming FDA which has by now become generally accepted.

Let us also recall that the classification and the explicit construction of FDA.s relies on two structural theorems by Sullivan showing how all possible such algebras are *cohomological extensions* of normal Lie algebras or superalgebras (for a recent and short review of these concepts just adapted to our purposes see [8]).

The Free Differential algebra of type IIA supergravity in $\mathrm{D} = 10$ can be obtained by dimensional reduction on a circle $\mathbb{S}^1$ from the FDA of the $\mathrm{D} = 11$ supergravity [23, 24]. Although straightforward this construction was never shown in the literature and it is quite lengthy and laborious. For this reason in appendix B we sketch the main steps of such a derivation. Furthermore, as we explain extensively in the sequel, our main target is the rheonomic parametrization of the FDA curvatures in the string frame and not in the Einstein frame. Hence in the quoted appendix we develop a mixed strategy to obtain our goal. We begin by constructing the rheonomic parametrization of the bosonic curvatures in the Einstein frame using dimensional reduction from D=11. In this way we also obtain the bosonic field equations of type IIA supergravity from dimensional reduction which is an easier task than deriving them from the Bianchi identities or from the construction of the $D = 10$ action. Next we perform a Weyl transformation to the string frame which changes the bosonic field equations only by an easy rescaling and once we have obtained the rheonomic parametrizations of the FDA curvatures in the string frame we directly determine the rheonomic parametrization of the fermionic curvatures in that frame from the analysis of the Bianchi identities.

All the above mentioned steps are discussed in the appendix. In the main text, we simply present the final result, namely type IIA supergravity in the string frame.

2.1 Type IIA FDA in the string frame

The field content of type IIA supergravity is given in table 2. This field content corresponds to the basic forms of a specific Free Differential Algebra including the 0-form items entering the rheonomic parametrizations of its curvatures.

The starting point is, as usual, the superPoincaré algebra. In $\mathrm{D} = 10$ we have two superPoincaré algebras with 32 supercharges, the type IIA and the type IIB. The Maurer Cartan description of the type IIA superalgebra is obtained by setting to zero the following curvatures:

Form/degree	string sector	SO(1,9)-rep/Chirality	superstring zero modes
V^a - [1]	NS-NS	$(2,0,0,0,0)$	graviton $h_{\mu\nu}$
ψ_R - [1]	R-NS	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ - right	gravitino $\psi_{R\mu}$
ψ_L - [1]	NS-R	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ - left	gravitino $\psi_{L\mu}$
$\mathbf{B}^{[2]}$ - [2]	NS-NS	$(1,1,0,0,0)$	Kalb-Ramond
$\mathbf{C}^{[1]}$ - [1]	R-R	$(1,0,0,0,0)$	R-R 1-form
$\mathbf{C}^{[3]}$ - [3]	R-R	$(1,1,1,0,0)$	R-R 3-form
χ_R	NS-R	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$- right	dilatino right
χ_L	R-NS	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$- left	dilatino left
φ	NS-NS	$(0,0,0,0,0)$	dilaton

Table 2: Field content of type IIA supergravity.

Type IIA superPoicaré algebra in the string frame.

$$R^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \tag{2.1}$$

$$T^a \equiv \mathcal{D} V^a - \mathrm{i}\frac{1}{2}\left(\overline{\psi}_L \wedge \Gamma^a \psi_L + \overline{\psi}_R \wedge \Gamma^a \psi_R\right) \tag{2.2}$$

$$\rho_{L,R} \equiv \mathcal{D}\psi_{L,R} \equiv d\psi_{L,R} - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\,\psi_{L,R} \tag{2.3}$$

$$\mathbf{G}^{[2]} \equiv d\mathbf{C}^{[1]} + \exp[-\varphi]\,\overline{\psi}_R \wedge \psi_L \tag{2.4}$$

$$\mathbf{f}^{[1]} \equiv d\varphi \tag{2.5}$$

$$\nabla\chi_{L/R} \equiv d\chi_{L,R} - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\,\chi_{L,R} \tag{2.6}$$

where the 0-form dilaton φ appearing in eq. (2.4) introduces a mobile coupling constant. Furthermore, V^a, ω^{ab} denote the vielbein and the spin connection 1-forms, respectively, while the two fermionic 1-forms $\psi_{L/R}$ are Majorana-Weyl spinors of opposite chirality:

$$\Gamma_{11}\,\psi_{L/R} = \pm\,\psi_{L/R}\,. \tag{2.7}$$

The flat metric $\eta_{ab} = \mathrm{diag}(+,-,\ldots,-)$ is the mostly minus one and Γ_{11} is hermitian and squares to the the identity $\Gamma_{11}^2 = \mathbf{1}$.

Setting $R^{ab} = T^a = \mathbf{G}^{[2]} = \mathbf{f}^{[1]} = 0$ one obtains the Maurer Cartan equations of a superalgebra where the spinor charges, $Q_{L,R}$ dual to the spinor 1-forms $\psi_{L,R}$ not only anticommute to the translations P_a but also to a central charge Z dual to the (Ramond Ramond) 1-form $\mathbf{C}^{[1]}$.

According to Sullivan's second theorem the FDA extension of the above superalgebra is dictated by its cohomology. In a first step one finds that there exists a cohomology class of degree three which motivates the introduction of a new 2-form generator $\mathbf{B}^{[2]}$ which in the superstring interpretation is just the Kalb-Ramond field. Considering then the cohomology of the FDA-extended algebra one finds a degree four cohomology class which motivates the introduction of a 3-form generator $\mathbf{C}^{[3]}$. In the superstring interpretation, this is just the second R-R field, the first being the gauge field $\mathbf{C}^{[1]}$. Altogether the complete type IIA FDA is obtained by adjoining the following curvatures to those already introduced:

The FDA extension of the type IIA superalgebra in the string frame.

$$\begin{aligned}
\mathbf{H}^{[3]} &= d\mathbf{B}^{[2]} + \mathrm{i}\left(\overline{\psi}_L \wedge \Gamma_a \psi_L - \overline{\psi}_R \wedge \Gamma_a \psi_R\right) \wedge V^a && (2.8)\\
\mathbf{G}^{[4]} &= d\mathbf{C}^{[3]} + \mathbf{B}^{[2]} \wedge d\mathbf{C}^{[1]} \\
&\quad - \frac{1}{2}\exp\left[-\varphi\right]\left(\overline{\psi}_L \wedge \Gamma_{ab}\psi_R + \overline{\psi}_R \wedge \Gamma_{ab}\psi_L\right) \wedge V^a \wedge V^b\,. && (2.9)
\end{aligned}$$

Equations (2.1)–(2.5) together with eqs. (2.8)–(2.9) provide the complete definition of the type IIA Free Differential Algebra.

The next task is that of writing the Bianchi identities and construct their rheonomic solution.

The Bianchi identities. The curvature definitions listed above lead immediately to the following Bianchi identities which we write, already under the assumption that the torsion is zero $T^a = 0$:

$$\begin{aligned}
0 &= \mathcal{D}\, R^{ab} && (2.10)\\
0 &= R^{ab} \wedge V_b - \mathrm{i}\left(\overline{\psi}_L \wedge \Gamma^a \rho_L + \overline{\psi}_R \wedge \Gamma^a \rho_R\right) && (2.11)\\
0 &= \mathcal{D}\,\rho_{L/R} + \frac{1}{4} R^{ab} \wedge \Gamma_{ab}\,\psi_{L/R} && (2.12)\\
0 &= d\,\mathbf{G}^{[2]} + \mathbf{f}^{[1]} \wedge \exp[-\varphi]\,\overline{\psi}_R \wedge \psi_L + \exp[-\varphi]\left(\overline{\psi}_R \wedge \rho_L - \overline{\psi}_L \wedge \rho_R\right) && (2.13)\\
0 &= d\mathbf{f}^{[1]} && (2.14)\\
0 &= d\,\mathbf{H}^{[3]} + 2\,\mathrm{i}\left(\overline{\psi}_L \wedge \Gamma_a \rho_L - \overline{\psi}_R \wedge \Gamma_a \rho_R\right) \wedge V^a && (2.15)\\
0 &= d\,\mathbf{G}^{[4]} - \mathbf{H}^{[3]} \wedge \mathbf{G}^{[2]} + \mathrm{i}\left(\overline{\psi}_L \wedge \Gamma_a \psi_L - \overline{\psi}_R \wedge \Gamma_a \psi_R\right) \wedge V^a \wedge \mathbf{G}^{[2]} \\
&\quad + \mathbf{H}^{[3]} \wedge \exp\left[-\varphi\right]\,\overline{\psi}_R \wedge \psi_L \\
&\quad - \frac{1}{2}\mathbf{f}^{[1]} \wedge \exp\left[-\varphi\right]\left(\overline{\psi}_L \wedge \Gamma_{ab}\psi_R + \overline{\psi}_R \wedge \Gamma_{ab}\psi_L\right) \wedge V^a \wedge V^b \\
&\quad - \exp\left[-\varphi\right]\left(\overline{\psi}_L \wedge \Gamma_{ab}\rho_R + \overline{\psi}_R \wedge \Gamma_{ab}\rho_L\right) \wedge V^a \wedge V^b && (2.16)\\
0 &= \mathcal{D}^2\,\chi_{L/R} + \frac{1}{4} R^{ab} \wedge \Gamma_{ab}\,\chi_{L/R}\,. && (2.17)
\end{aligned}$$

As it is the case for all supergravities and for all FDA.s the above Bianchi identities admit a unique rheonomic solution up to field redefinitions. The rheonomic solution of the Bianchis implies also the field equations of the theory given as a set of constraints to be satisfied by the space-time curvature components. The choice of a frame is performed by imposing an additional condition which fixes the field redefinitions. In particular we define the string frame by requiring both the vanishing of the torsion

$$T^a = 0 \tag{2.18}$$

and the vanishing of all of the fermionic sectors of the 3-form curvature $\mathbf{H}^{[3]}$. This amounts to setting:

$$\mathbf{H}^{[3]} = \mathcal{H}_{abc}\, V^a \wedge V^b \wedge V^c\,. \tag{2.19}$$

One can indeed verify that the fulfillment of the above conditions requires a Weyl rescaling of the fields which yield the usual prefactor $e^{-2\varphi}$ in front of the NS-NS and the fermionic sector of the action. The relevance of the frame-fixing choice (2.19) will be illustrated in section 3 where we discuss the Green-Schwarz superstring action and κ-symmetry.

2.2 Rheonomic parametrizations of the type IIA curvatures in the string frame

In order to present our result in the form most suitable to our later purposes, namely the discussion of the BRST chiral algebra which leads to the construction of the pure spinor superstring action, it is convenient to introduce a set of tensors, which involve both the supercovariant field strengths $\mathcal{G}_{ab}, \mathcal{G}_{abcd}$ of the Ramond-Ramond p-forms and also bilinear currents in the dilatino field $\chi_{L/R}$. The needed tensors are those listed below:

$$\begin{aligned}
\mathcal{M}_{ab} &= \left(\frac{1}{8}\exp[\varphi]\,\mathcal{G}_{ab} + \frac{9}{64}\,\overline{\chi}_R\,\Gamma_{ab}\,\chi_L\right)\\
\mathcal{M}_{abcd} &= -\frac{1}{16}\exp[\varphi]\,\mathcal{G}_{abcd} - \frac{3i}{256}\,\overline{\chi}_L\,\Gamma_{abcd}\,\chi_R\\
\mathcal{N}_0 &= \frac{3}{4}\,\overline{\chi}_L\,\chi_R\\
\mathcal{N}_{ab} &= \frac{1}{4}\exp[\varphi]\,\mathcal{G}_{ab} + \frac{9}{32}\,\overline{\chi}_R\,\Gamma_{ab}\,\chi_L = 2\,\mathcal{M}_{ab}\\
\mathcal{N}_{abcd} &= \frac{1}{24}\exp[\varphi]\,\mathcal{G}_{abcd} + \frac{1}{128}\,\overline{\chi}_R\,\Gamma_{abcd}\,\chi_L = -\frac{2}{3}\mathcal{M}_{abcd}\,.
\end{aligned} \tag{2.20}$$

The above tensors are conveniently assembled into the following spinor matrices

$$\mathcal{Z} = \mathcal{N}_{ab}\Gamma^{ab} + 3\,\mathcal{N}_{abcd}\,\Gamma^{abcd} \tag{2.21}$$

$$\mathcal{M}_{\pm} = \mathrm{i}\left(\mp\mathcal{M}_{ab}\,\Gamma^{ab} + \mathcal{M}_{abcd}\,\Gamma^{abcd}\right) \tag{2.22}$$

$$\mathcal{N}^{(even)}_{\pm} = \mp\mathcal{N}_0\,\mathbf{1} + \mathcal{N}_{ab}\,\Gamma^{ab} \mp \mathcal{N}_{abcd}\,\Gamma^{abcd} \tag{2.23}$$

$$\mathcal{N}^{(odd)}_{\pm} = \pm\frac{i}{3}f_a\Gamma^a \pm \frac{1}{64}\overline{\chi}_{R/L}\Gamma_{abc}\chi_{R/L}\Gamma^{abc} - \frac{i}{12}\mathcal{H}_{abc}\Gamma^{abc} \tag{2.24}$$

$$\mathcal{L}^{(odd)}_{a\,\pm} = \mathcal{M}_{\mp}\,\Gamma_a\,; \qquad \mathcal{L}^{(even)}_{a\,\pm} = \mp\frac{3}{8}\,\mathcal{H}_{abc}\,\Gamma^{bc}\,. \tag{2.25}$$

In terms of these objects the rheonomic parametrizations of the curvatures, solving the Bianchi identities can be written as follows:

Bosonic curvatures.

$$T^a = 0 \tag{2.26}$$

$$\begin{aligned}
R^{ab} &= R^{ab}{}_{mn}\,V^m \wedge V^n + \overline{\psi}_R\,\Theta^{ab}_{m|L} \wedge V^m + \overline{\psi}_L\,\Theta^{ab}_{m|R} \wedge V^m\\
&\quad + \mathrm{i}\,\frac{3}{4}\left(\overline{\psi}_L \wedge \Gamma_c\,\psi_L - \overline{\psi}_R \wedge \Gamma_c\,\psi_R\right)\mathcal{H}^{abc}\\
&\quad + \overline{\psi}_L \wedge \Gamma^{[a}\,\mathcal{Z}\,\Gamma^{b]}\,\psi_R
\end{aligned} \tag{2.27}$$

$$\mathbf{H}^{[3]} = \mathcal{H}_{abc}V^a \wedge V^b \wedge V^c \tag{2.28}$$

$$\mathbf{G}^{[2]} = \mathcal{G}_{ab}V^a \wedge V^b + \mathrm{i}\,\frac{3}{2}\exp[-\varphi]\left(\overline{\chi}_L\,\Gamma_a\,\psi_L + \overline{\chi}_R\,\Gamma_a\,\psi_R\right) \wedge V^a \tag{2.29}$$

$$\mathbf{f}^{[1]} = f_a V^a + \frac{3}{2}\left(\overline{\chi}_R\,\psi_L - \overline{\chi}_L\,\psi_R\right) \tag{2.30}$$

$$\begin{aligned}
\mathbf{G}^{[4]} &= \mathcal{G}_{abcd}V^a \wedge V^b \wedge V^c \wedge V^d\\
&\quad - \mathrm{i}\,\frac{1}{2}\exp[-\varphi]\left(\overline{\chi}_L\,\Gamma_{abc}\,\psi_L - \overline{\chi}_R\,\Gamma_{abc}\,\psi_R\right) \wedge V^a \wedge V^b \wedge V^c\,.
\end{aligned} \tag{2.31}$$

Fermionic curvatures.

$$\rho_{L/R} = \rho_{ab}^{L/R} V^a \wedge V^b + \mathcal{L}_{a\pm}^{(even)} \psi_{L/R} \wedge V^a + \mathcal{L}_{a\mp}^{(odd)} \psi_{R/L} \wedge V^a + \rho_{L/R}^{(0,2)} \tag{2.32}$$

$$\nabla \chi_{L/R} = \mathcal{D}_a \chi_{L/R} V^a + \mathcal{N}_{\pm}^{(even)} \psi_{L/R} + \mathcal{N}_{\mp}^{(odd)} \psi_{R/L} . \tag{2.33}$$

Note that the components of the generalized curvatures along the bosonic vielbeins do not coincide with their spacetime components, but rather with their supercovariant extension. Indeed expanding for example the four-form along the spacetime differentials one finds that

$$\begin{aligned}\widetilde{G}_{\mu\nu\rho\sigma} \equiv \mathcal{G}_{abcd} V_\mu^a \wedge V_\nu^b \wedge V_\rho^c \wedge V_\sigma^d = \partial_{[\mu} C^{[4]}_{\nu\rho\sigma]} + B^{[2]}_{[\mu\nu} \partial_\rho C^{[1]}_{\sigma]} - \\ -\frac{1}{2} e^{-\varphi} \left(\overline{\psi}_{L[\mu} \Gamma_{\nu\rho} \psi_{R\sigma]} + \overline{\psi}_{R[\mu} \Gamma_{\nu\rho} \psi_{L\sigma]}\right) \\ + \mathrm{i}\,\frac{1}{2} \exp[-\varphi] \left(\overline{\chi}_L \Gamma_{[\mu\nu\rho} \psi_{L\sigma]} - \overline{\chi}_R \Gamma_{[\mu\nu\rho} \psi_{R\sigma]}\right)\end{aligned}$$

where $\widetilde{G}$ is the supercovariant field strength. In the parametrization (2.27) of the Riemann tensor we have used the following definition:

$$\Theta_{ab|cL/R} = -i\Big(\Gamma_a \rho_{bcR/L} + \Gamma_b \rho_{caR/L} - \Gamma_c \rho_{abR/L}\Big) . \tag{2.34}$$

Finally by $\rho_{L/R}^{(0,2)}$ we have denoted the fermion-fermion part of the gravitino curvature whose explicit expression can be written in two different forms, equivalent by Fierz rearrangement:

$$\begin{aligned}\rho_{L/R}^{(0,2)} &= \pm \frac{21}{32} \Gamma_a \chi_{R/L} \bar{\psi}_{L/R} \wedge \Gamma^a \psi_{L/R} \\ &\mp \frac{1}{2560} \Gamma_{a_1a_2a_3a_4a_5} \chi_{R/L} \left(\overline{\psi}_{L/R} \Gamma^{a_1a_2a_3a_4a_5} \psi_{L/R}\right)\end{aligned} \tag{2.35}$$

or

$$\rho_{L/R}^{(0,2)} = \pm \frac{3}{8} \mathrm{i}\, \psi_{L/R} \wedge \bar{\chi}_{R/L} \psi_{L/R} \pm \frac{3}{16} \mathrm{i}\, \Gamma_{ab} \psi_{L/R} \wedge \bar{\chi}_{R/L} \Gamma^{ab} \psi_{L/R} . \tag{2.36}$$

2.3 Comments on the curvature structure in the string frame

The rheonomic parametrizations presented in the previous section have some distinctive features which are deprived of any relevance in a supergravity context while they turn out to be crucial for the successful construction of a BRST invariant pure spinor superstring σ-model. Let us point these features out:

1. The rheonomic parametrization of the Neveu-Schwarz curvature $\mathbf{H}^{[3]}$ is purely inner, namely there are no dilatino terms on the right hand side. As we anticipated this is the very definition of the string frame and it is important in order to write a κ-symmetric Green-Schwarz superstring action.

2. The $(1,1)$ sector of the gravitino curvature $\rho_{L/R}^{(1,1)}$ is divided in two parts, one of the same chirality, which involves only Neveu-Schwarz field strengths and one of the opposite chirality which involves Ramond-Ramond field strengths instead:

$$\rho_{L/R}^{(1,1)} = \mp \underbrace{\frac{3}{8} \mathcal{H}_{abc} \Gamma^{ab} \psi_{L/R} \wedge V^c}_{\text{NS same chirality}} + \underbrace{\mathcal{M}_{\pm} \Gamma_a \psi_{R/L} \wedge V^a}_{\text{RR opposite chirality}} . \tag{2.37}$$

From a supergravity viewpoint we simply expect a linear combination of gamma matrices with coefficients given by the bosonic field strengths and the specific form of such a linear combination has no particular relevance. On the other hand, for the construction of a pure spinor BRST invariant action, the particular structure of $\rho^{(1,1)}_{L/R}$ in the mixed chirality sector, which singles out a matrix $\mathcal{M}_\pm$ with no vector indices, is just essential. Indeed, as we are going to see, the matrix $\mathcal{M}_+$ is just what can be used to introduce into the BRST Lagrangian a term of the form:

$$\overline{\mathbf{d}}_+ \mathcal{M}_+ \mathbf{d}_- \, e^+ \wedge e^-$$

the fields $\mathbf{d}_\pm$ being the Lagrange multipliers of the BRST complex. Such a term is the vertex operator of the Ramond-Ramond fields and it is an important part of Berkovits' construction. It would not be allowed if the Lorentz structures appearing in $\rho^{(1,1)}_{L/R}$ were different. It is remarkable that such a specific Lorentz structure, essential for the pure spinor part of the superstring action, appears precisely in the string frame, in which the Green-Schwarz part of the same superstring action is naturally formulated. For instance in the Einstein frame the Lorentz structures appearing in $\rho^{(1,1)}_{L/R}$ are different.

3. The $\rho^{(0,2)}_{L/R}$ part of the gravitino curvature is such that, also in the presence of general backgrounds, with non trivial dilatino fields, the contribution to $\rho^{(0,2)}_{L}$ is only from bilinears in ψ_L and that to $\rho^{(0,2)}_{R}$ is only from bilinears in ψ_R. This feature is apparent in both the expressions of $\rho^{(0,2)}_{L/R}$ given in (2.36) and will turn out to be crucial in proving the BRST invariance of the Berkovits action since it implies that the anticommutator of the left-handed BRST operator with the right handed one vanishes on the gravitino field. It is once again remarkable that this third essential feature of the rheonomic parametrizations occurs in the same frame as the other two. Indeed the mentioned structure of $\rho^{(0,2)}_{L/R}$ is not true in the Einstein frame.

The above discussion has been anticipated in order to emphasize that the subsequent construction of a Berkovits-like pure spinor superstring action is just founded on the existence of a supergravity string frame where the rheonomic parametrizations display the three features mentioned above. In solving the Bianchi identities it is by no means obvious a priori that these features should simultaneously appear. Yet they do and this gives rise to the Berkovits sigma model.

2.4 Field equations of type IIA supergravity in the string frame

As usual the rheonomic parametrizations of the supercurvatures imply, via Bianchi identities a certain number of constraints on the inner components of the same curvatures which can be recognized as the field equations of type IIA supergravity. We derived the bosonic part of these field equations in two steps: First we performed the Einstein frame dimensional reduction on a circle of the field equations of D = 11 supergravity. Then we applied the Weyl transformation which relates the Einstein frame to the string frame:

$$V^a_{(E)} = V^a_{(S)} e^{-\varphi/4} \,. \tag{2.38}$$

Obviously we could have obtained the same result directly from the Bianchi identities in the string frame, yet this would have been much more laborious.

The result is the following one. We have an Einstein equation of the following form:

$$\mathcal{R}_{ab} = \widehat{T}_{ab}(f) + \widehat{T}_{ab}(\mathcal{G}_2) + \widehat{T}_{ab}(\mathcal{H}) + \widehat{T}_{ab}(\mathcal{G}_4) \tag{2.39}$$

where the stress-energy tensor on the right hand side are defined as

$$\widehat{T}_{ab}(f) = -\mathcal{D}_a\mathcal{D}_b\varphi + \frac{8}{9}\mathcal{D}_a\varphi\,\mathcal{D}_b\varphi - \eta_{ab}\left(\frac{1}{6}\Box\varphi + \frac{5}{9}\mathcal{D}^m\varphi\,\mathcal{D}_m\varphi\right) \tag{2.40}$$

$$\widehat{T}_{ab}(\mathcal{G}_2) = \exp[2\varphi]\,\mathcal{G}_{ax}\,\mathcal{G}_{by}\,\eta^{ab} \tag{2.41}$$

$$\widehat{T}_{ab}(\mathcal{H}) = -\exp\left[\frac{1}{3}\varphi\right]\left(\frac{9}{8}\mathcal{H}_{axy}\,\mathcal{H}_{bwt}\,\eta^{xw}\,\eta^{yt} - \frac{1}{8}\eta_{ab}\,\mathcal{H}_{xyz}\,\mathcal{H}^{xyz}\right) \tag{2.42}$$

$$\widehat{T}_{ab}(\mathcal{G}_4) = \exp[2\varphi]\left(6\mathcal{G}_{ax_1x_2x_3}\mathcal{G}_{by_1y_2y_3}\eta^{x_1y_1}\eta^{x_2y_2}\eta^{x_3y_3} - \frac{1}{2}\eta_{ab}\mathcal{G}_{x_1\dots x_4}\mathcal{G}^{x_1\dots x_4}\right). \tag{2.43}$$

Next we have the equations for the dilaton and the Ramond 1-form:

$$\begin{aligned} 0 &= \Box\varphi - 2f_a f^a + \frac{3}{2}\exp[2\varphi]\,\mathcal{G}^{x_1x_2}\,\mathcal{G}_{x_1x_2} \\ &\quad + \frac{3}{2}\exp[2\varphi]\,\mathcal{G}^{x_1x_2x_3x_4}\,\mathcal{G}_{x_1x_2x_3x_4} + \frac{3}{4}\exp\left[\frac{4}{3}\varphi\right]\mathcal{H}^{x_1x_2x_3}\,\mathcal{H}_{x_1x_2x_3} \end{aligned} \tag{2.44}$$

$$0 = \mathcal{D}_m\,\mathcal{G}^{ma} - \frac{5}{3}f^m\,\mathcal{G}_{ma} + 3\,\mathcal{G}^{ax_1x_2x_3}\,\mathcal{H}_{x_1x_2x_3} \tag{2.45}$$

and the equations for the NS 2-form and for the RR 3-form:

$$\begin{aligned} 0 &= \mathcal{D}_m\,\mathcal{H}^{mab} - \frac{2}{3}f^m\,\mathcal{H}_{mab} \\ &\quad - \exp\left[\frac{4}{3}\varphi\right]\left(4\,\mathcal{G}^{x_1x_2ab}\,\mathcal{G}_{x_1x_2} - \frac{1}{24}\epsilon^{abx_1\dots x_8}\,\mathcal{G}_{x_1x_2x_3x_4}\,\mathcal{G}_{x_5x_6x_7x_8}\right) \end{aligned} \tag{2.46}$$

$$\begin{aligned} 0 &= \mathcal{D}_m\,\mathcal{G}^{ma_1a_2a_3} + \frac{1}{3}f_m\,\mathcal{G}^{ma_1a_2a_3} \\ &\quad + \exp\left[\frac{2}{3}\varphi\right]\left(\frac{3}{2}\mathcal{G}^{m[a_1}\,H^{a_2a_3]n}\,\eta_{mn} + \frac{1}{48}\epsilon^{a_1a_2a_3x_1\dots x_7}\mathcal{G}_{x_1x_2x_3x_4}\,H_{x_5x_6x_7}\right). \end{aligned} \tag{2.47}$$

Any solution of these bosonic set of equations can be uniquely extended to a full superspace solution involving 32 theta variables by means of the rheonomic conditions. The implementation of such a fermionic integration is the *supergauge completion*.

In this way we have completed the discussion of type IIA supergravity in the string frame. Let us now turn to superstrings.

3. The Green-Schwarz action and κ-symmetry

As we already mentioned in the introduction the Green-Schwarz κ-symmetric action of type II superstrings has exactly the same form (in the string frame of background supergravity fields) for the IIA and IIB case. It is just the form of the κ-symmetry transformation against which it is invariant that is slightly different in the two cases. Actually also these transformations are essentially the same up to the obvious replacement of $\psi_{L/R}$ gravitinos with their chiral $\psi_{1,2}$ analogues and similarly for the parameters.

The reason for this equality of the type IIA and type IIB actions is due to the following peculiarity which characterizes both the NS and the GS superstring formalism: they introduce into the action just one half of the (super)-forms describing (super)-space geometry. All the well known difficulties connected with the description of RR emission vertices and with the string quantization in non-trivial RR backgrounds are connected to this blindness of the formalism which ignores half of the geometry. The new BRST formulation of superstring actions based on pure-spinor superghosts is the only, so far discovered, way-out of this contradiction. Indeed in the pure spinor approach the ghost-antighost sector appears to provide the missing fields which couple to the other face of the moon, namely the fermionic forms $\psi_{L/R}$ and the RR superforms. Hence the BRST invariant actions of type IIA and type IIB theory will be different although similar just as the κ-symmetry transformations are slightly different in the two cases. The BRST form of the action is just an extension of the Green-Schwarz action which is identical in the two cases. This is the world sheet counterpart of what happens for the bulk supergravity action. Also there restricting the Lagrangian to the Neveu-Schwarz sector we obtain identical sub-Lagrangians while it is the extension by means of the fermionic and Ramond Ramond fields that is different in the two cases A and B.

In this section we construct the Green-Schwarz action of type II superstrings moving in a generic supergravity background and we consider its invariance against κ-symmetry in the case of type IIA superstrings.

3.1 The general form of the GS action

Employing, as it is required by the rheonomic construction of the Lagrangian, the first order formalism [25], we write the Green-Schwarz action as the sum of two addenda, the kinetic and the Wess-Zumino contributions:

$$\mathcal{A}_{\text{GS}} = \mathcal{A}_{\text{kin}} + A_{\text{WZ}} \tag{3.1}$$

where:

$$\begin{aligned} \mathcal{A}_{\text{kin}} &= \int \Big(\Pi^a_+ \, V^b \, \eta_{ab} \wedge e^+ - \Pi^a_- \, V^b \, \eta_{ab} \wedge e^- \\ &\qquad\qquad + \frac{1}{2} \Pi^a_i \, \Pi^b_j \, \eta^{ij} \, \eta_{ab} \, e^+ \wedge e^- \Big) \end{aligned} \tag{3.2}$$

$$A_{\text{WZ}} = \frac{1}{2} \, q \int \mathbf{B}^{[2]} \,. \tag{3.3}$$

In the above two formulae, $e^\pm = e^0 \pm e^1$ denote the zweibein of the string world-sheet in light-cone basis for the $2d$ Lorentz indices, namely $\eta_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, while by $\Pi^a_\pm$ we have denoted the usual 0-form auxiliary field whose equation identifies it with the projection of the target vielbein V^a onto the world volume zweibein $e^\pm$. The coefficient q, denoting the string charge, is fixed in such a way as to obtain a completely κ supersymmetric action in any background. As we already stressed there is no dilaton prefactor in the above action since the FDA gauge forms (in particular the vielbein) and curvatures were already transformed to the string frame.

First of all let us check the relative coefficients in the kinetic action (3.2) by calculating its variation with respect to the auxiliary field $\Pi^a_\pm$. We obtain:

$$\begin{aligned} 0 = \frac{\delta\,\mathcal{A}_{\rm kin}}{\delta \Pi^a_\pm} &= \int \left(\pm\,\eta_{ab}\,V^b \wedge e^\pm + \eta_{ab}\,\Pi^b_\pm\, e^+ \wedge e^-\right) \\ \Downarrow \\ V^a = \Pi^a_+\, e^+ &+ \Pi^a_-\, e^- \end{aligned} \tag{3.4}$$

which is the required result for the elimination of the auxiliary field Π^a_- and the transition to second order formalism.

Next let us introduce the following short hand notation:

$$\mathbf{\Gamma}_\pm \equiv \Pi^a_\pm\,\Gamma_a \tag{3.5}$$

and let us check the κ-symmetric invariance of the GS action in the A case.

3.2 κ-symmetry in the type IIA case

Relying on the rheonomic parametrizations of the FDA let us calculate the variation of the Green-Schwarz action (3.1) under a target supersymmetry of parameters $\epsilon_{L/R}$. We obtain:

$$\begin{aligned} \delta_{\rm susy}\,\mathcal{A}_{\rm kin} = \int \mathrm{i}\big[\,&(\bar\epsilon_L\,\mathbf{\Gamma}_+\,\psi_L + \bar\epsilon_R\,\mathbf{\Gamma}_+\,\psi_R) \wedge e^+ \\ &- (\bar\epsilon_L\,\mathbf{\Gamma}_-\,\psi_L + \bar\epsilon_R\,\mathbf{\Gamma}_-\,\psi_R) \wedge e^-\,\big] \end{aligned} \tag{3.6}$$

$$\begin{aligned} \delta_{\rm susy}\,\mathcal{A}_{\rm WZ} = -\,q\int \mathrm{i}\big[\,&(\bar\epsilon_L\,\mathbf{\Gamma}_+\,\psi_L - \bar\epsilon_R\,\mathbf{\Gamma}_+\,\psi_R) \wedge e^+ \\ &+ (\bar\epsilon_L\,\mathbf{\Gamma}_-\,\psi_L - \bar\epsilon_R\,\mathbf{\Gamma}_-\,\psi_R) \wedge e^-\,\big]\,. \end{aligned} \tag{3.7}$$

Let us now recall that the rules of the 1.5-order formalism which we use in all our p-brane constructions impose that, after variation, we should implement the field equations of all the auxiliary fields whose equation of motion is algebraic and allows for their own elimination in terms of dynamical fields. In the string action these latter are the 0-form fields Π^a_i and the 2-dimensional zweibein e^i. The field equation of the first is (3.4) while the field equation of the zweibein is simply:

$$\eta_{ab}\,\Pi^a_i\,\Pi^b_j = \eta_{ij} \tag{3.8}$$

namely the statement that the world-sheet metric is the pull-back of the target superspace metric. Under these conditions one obtains:

$$\mathcal{L}^{(0)}_{\rm kin} = -\,e^+ \wedge e^- \tag{3.9}$$

where:

$$\mathcal{L}^{(0)}_{\rm kin} \equiv \left(\Pi^a_+\,V^b\,\eta_{ab} \wedge e^+ - \Pi^a_-\,V^b\,\eta_{ab} \wedge e^- + \frac{1}{2}\Pi^a_i\,\Pi^b_j\,\eta^{ij}\,\eta_{ab}\,e^+ \wedge e^-\right) \tag{3.10}$$

is the 2-form which corresponds to the kinetic Lagrangian.

Assembling these results we find:

$$\delta_{\text{susy}}\,\mathcal{A}_{\text{GS}} = \int \mathrm{i}\,[\,(1-q)\,\overline{\epsilon_L}\,\mathbf{\Gamma}_+\psi_L \,+\, (1+q)\,\overline{\epsilon_R}\,\mathbf{\Gamma}_+\psi_R]\wedge e^+ \\ -\,\mathrm{i}\,[(1+q)\,\overline{\epsilon_L}\,\mathbf{\Gamma}_-\psi_L \,+\, (1-q)\,\overline{\epsilon_R}\,\mathbf{\Gamma}_-\psi_R]\wedge e^- \,. \tag{3.11}$$

The above variation vanishes under the following conditions:

$$\begin{aligned} q &= 1 \\ \overline{\epsilon}_L &= \overline{\epsilon}_L\,P^+ \\ \overline{\epsilon}_R &= \overline{\epsilon}_R\,P^- \end{aligned} \tag{3.12}$$

where:

$$P^{\pm} = \frac{1}{2}(1 \pm \mathbf{\Gamma}_{+-}) \tag{3.13}$$

is the κ supersymmetry projector. Indeed we have $P^{\pm}\,\mathbf{\Gamma}_{\mp} = 0$ and $P^{\pm}\,\mathbf{\Gamma}_{+-} = 1$ which are the necessary and sufficient conditions in order for half of the terms in eq. (3.11) to cancel. The other half of them cancel thanks to the choice of the parameter q.

This concludes the derivation of the κ-symmetric action of a type IIA superstring moving in the background of any supergravity solution, namely of any solution of the type IIA field equations lifted to the whole $(10,32)$-dimensional superspace by means of rheonomy. The above formulae encode a complete algorithm to write down the explicit Green-Schwarz bosonic-fermionic sigma model once the explicit form of the superforms V^a, $\mathbf{B}^{[2]}$, φ is given. However, since the most general background is characterized by mutually interacting fermion, NS-NS and R-R fields, these latter have to be determined at the same time as the NS-NS forms and the fermionic gravitino forms $\psi_{L/R}$.

3.3 Background independence

It should be stressed that both in the case of the Green-Schwarz actions or of their descendant Pure Spinor actions the problem of constructing the sigma model is always split into two conceptually well separated parts:

(a) Construction of the action in a generic supergravity FDA background;

(b) Super-gauge completion, namely explicit integration of the rheonomic conditions in a given bosonic background in order to produce the explicit θ-dependence of the superforms appropriate to that background.

The solution of point (a) is universal, can be done once for all and it is the goal of the present paper. Point (b) is obviously case dependent and can be more or less technically difficult depending on the structure of the chosen background. Yet it must be observed that it is a guaranteed step since the fermionic equations to be integrated are of the first order and integrable by very construction. The issue is just a matter of elegance and brevity in writing the solution, which can always be reached, although in most cases its explicit

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expression may require a considerable calculational effort. We stress this fact because there has been some confusion about this in the literature, particularly in connection with the pure spinor formulation. The pure spinor σ-model has been constructed case by case on given backgrounds as if the form of the action and the BRST transformations had to be reinvented each time. This has probably somehow obscured the general structure and the remarkable economy of principles underlying this new setup which solves some of the open questions in superstring quantization.

4. The pure spinor action and BRST symmetry

As advocated at the end of the previous section the alternative to κ-symmetry is the BRST quantization of the Green-Schwarz action by means of constrained BRST transformations using pure spinors superghosts. The procedure consists of the following three steps:

(a) Derivation of the constrained BRST algebra in the non-negative ghost-number sector from the FDA curvatures and their rheonomic parametrizations;

(b) Introduction of antighosts $w^{\pm}$ and Lagrange multipliers $\mathbf{d}^{\pm}$ whose BRST transformation is defined up to a new gauge symmetry;

(c) Construction of a *gauge fixing* action $\mathcal{A}_{\text{gf}}$ to be added to the classical Green-Schwarz action A_{GS} such that its variation under BRST cancels that of the classical action thanks to the non vanishing BRST variation of the Lagrange multipliers amounting to new gauge symmetries.

Let us begin with step (a).

4.1 The constrained BRST algebra from the FDA

Applying the general procedure we can obtain the explicit form of the constrained BRST algebra suitable for either the type IIA or the type IIB theory by performing the ghost-form extension of the Free Differential Algebra curvature definitions and parametrizations successively setting to zero the bosonic ghosts. Actually, once the principle has been clarified we can perform the two steps at once by considering the purely fermionic extension, namely:

$$\begin{aligned} \varphi &\mapsto \varphi \\ V^a &\mapsto V^a \\ \mathbf{B}^{[2]} &\mapsto \mathbf{B}^{[2]} \\ \mathbf{C}^{[1]} &\mapsto \mathbf{C}^{[1]} \\ \mathbf{C}^{[3]} &\mapsto \mathbf{C}^{[3]} \\ \psi_{L/R} &\mapsto \psi_{L/R} + \lambda_{L/R} \,. \end{aligned} \tag{4.1}$$

Each extended curvature definition $\widehat{\mathbf{R}}^{[p]}_{\text{def}}$ and each extended curvature parametrization $\widehat{\mathbf{R}}^{[p]}_{\text{par}}$ decomposes into ghost sectors according to:

$$\begin{aligned} \widehat{\mathbf{R}}^{[p]}_{\text{def}} &= \mathbf{R}^{[p,0]}_{\text{def}} + \mathbf{R}^{[p-1,1]}_{\text{def}} + \mathbf{R}^{[p-2,2]}_{\text{def}} \\ \widehat{\mathbf{R}}^{[p]}_{\text{par}} &= \mathbf{R}^{[p,0]}_{\text{par}} + \mathbf{R}^{[p-1,1]}_{\text{par}} + \mathbf{R}^{[p-2,2]}_{\text{par}} \end{aligned} \tag{4.2}$$

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where we stop at ghost number $g = 2$ since neither in the curvature definitions nor in the curvature parametrizations there appear higher than quadratic powers of the $\psi_{L/R}$ forms. Then we have to impose:

$$
\begin{aligned}
\mathbf{R}^{[p,0]}_{\rm def} &= \mathbf{R}^{[p,0]}_{\rm par} \\
\mathbf{R}^{[p-1,1]}_{\rm def} &= \mathbf{R}^{[p-1,1]}_{\rm par} \\
\mathbf{R}^{[p-2,2]}_{\rm def} &= \mathbf{R}^{[p-2,2]}_{\rm par} \,.
\end{aligned}
\tag{4.3}
$$

The first of eqs. (4.3) is simply the rheonomic parametrization of the classical curvature we started from. The second equation defines the constrained BRST transformation of all the physical fields. The last of eqs. (4.3) defines the BRST transformation of the ghost fields (the pure spinors) when the right hand side is non zero ($\mathbf{R}^{[p-2,2]}_{\rm par} \neq 0$) and the quadratic pure spinor constraints $\mathbf{R}^{[p-2,2]}_{\rm def} = 0$ when the right hand side is zero $\mathbf{R}^{[p-2,2]}_{\rm par} = 0$. Let us write the result of these straightforward manipulations.

4.2 The constrained BRST algebra of type IIA theories

It is also convenient to split the BRST operator into two chiral sectors. The BRST operator is written as:

$$
\mathcal{S} = \mathcal{S}_L + \mathcal{S}_R \tag{4.4}
$$

where $\mathcal{S}_{L/R}$ shifts in the direction of $\lambda_{L/R}$.

In this way from the $(p-1,1)$ sector we obtain the BRST chiral transformations of the physical fields:

$$
\begin{aligned}
\mathcal{S}_{L/R}\,\mathbf{B}^{[2]} &= \mp 2\,\mathrm{i}\,\overline{\psi}_{L/R}\,\Gamma_a\,\lambda_{L/R}\,V^a \\
\mathcal{S}_{L/R}\,\mathbf{C}^{[1]} &= \mp \exp[-\varphi]\,\overline{\psi}_{R/L}\,\lambda_{L/R} + \frac{3}{2}\,\mathrm{i}\,\exp[-\varphi]\,\overline{\chi}_{L/R}\,\Gamma_a\,\lambda_{L/R}\,V^a \\
\mathcal{S}_{L/R}\,\mathbf{C}^{[3]} &= \overline{\psi}_{R/L}\,\Gamma_{ab}\,\lambda_{L/R}\,V^a \wedge V^b - B^{[2]} \wedge \mathcal{S}_{L/R}C^{[1]} \\
&\quad \mp \mathrm{i}\,\frac{1}{2}\,\exp[-\varphi]\,\overline{\chi}_{L/R}\,\Gamma_{abc}\,\lambda_{L/R}\,V^a \wedge V^b \wedge V^c \\
\mathcal{S}_{L/R}\,V^a &= \mathrm{i}\,\overline{\psi}_{L/R}\,\Gamma^a\,\lambda_{L/R} \\
\mathcal{S}_{L/R}\psi_{L/R} &= -\mathcal{D}\,\lambda_{L/R} \mp \frac{3}{8}\,\Gamma^{a_1a_2}\,\lambda_{L/R}\,V^{a_3}\,\mathcal{H}_{a_1a_2a_3} \pm \frac{21}{16}\,\Gamma_a\chi_{R/L}\,(\overline{\psi}_{L/R}\,\Gamma^a\lambda_{L/R}) \\
&\quad \mp \frac{1}{1280}\,\Gamma_{a_1\dots a_5}\chi_{R/L}\,(\overline{\psi}_{L/R}\,\Gamma^{a_1\dots a_5}\lambda_{L/R}) \\
\mathcal{S}_{R/L}\psi_{L/R} &= \mathcal{M}_{\pm}\,\Gamma_b\lambda_{R/L}\,V^b
\end{aligned}
\tag{4.5}
$$

while from the sectors $(p-2,2)$ we obtain the transformation of the superghosts:

$$
\begin{aligned}
\mathcal{S}_{L/R}\lambda_{L/R} &= \pm\frac{21}{16}\,\Gamma_a\chi_{R/L}\,(\overline{\lambda}_{L/R}\,\Gamma^a\lambda_{L/R}) \\
&\quad \mp \frac{1}{1280}\,\Gamma_{a_1\dots a_5}\chi_{R/L}\,(\overline{\lambda}_{L/R}\,\Gamma^{a_1\dots a_5}\lambda_{L/R}) \\
\mathcal{S}_{R/L}\lambda_{L/R} &= 0
\end{aligned}
\tag{4.6}
$$

and the following *pure spinor constraints*:

$$
\begin{aligned}
0 &= \overline{\lambda}_L \, \Gamma_a \, \lambda_L \; + \; \overline{\lambda}_R \, \Gamma_a \, \lambda_R && (4.7)\\
0 &= \big(\overline{\lambda}_L \, \Gamma_a \, \lambda_L \; - \; \overline{\lambda}_R \, \Gamma_a \, \lambda_R\big) \wedge V^a && (4.8)\\
0 &= \exp[-\varphi] \, \big(\overline{\lambda}_R \, \lambda_L\big) && (4.9)\\
0 &= \exp[-\varphi] \, \overline{\lambda}_R \, \Gamma_{ab} \, \lambda_L \, V^a \wedge V^b \, . && (4.10)
\end{aligned}
$$

Before discussing the complete structure of the BRST transformations on the background fields as a consequence of the extension of the rheonomic parameterizations, we need to clarify how the constraints (4.7)–(4.10) have to be understood. It is clear that these constraints are too strong for a 10d target-space vielbein V^a and therefore we have to project them on the 2d surface by embedding the worldsheet into the target-space. In particular the vielbeins V^a must be replaced by the embedding rectangular matrices $\Pi^a_\pm$. As will be shown in a separate paper [26], the set of constraints given above are equivalent to the constraints given by [2]. This will be proven by showing that the solution of the constraints (4.7)–(4.10) gives 22 independent complex degrees of freedom.[1] [2]

Finally it is also necessary to write down the chiral BRST transformations of the dilatino field:

$$
\begin{aligned}
\mathcal{S}_{L/R} \, \chi_{L/R} &= \mathcal{N}^{(even)}_{\pm} \, \lambda_{L/R}\\
\mathcal{S}_{R/L} \, \chi_{L/R} &= \mathcal{N}^{(odd)}_{\mp} \, \lambda_{R/L} \, . && (4.11)
\end{aligned}
$$

Let us give, for the sake of completeness, the formulas defining the action of the BRST operator on the field strengths, some of which will be needed in the final section

$$
\begin{aligned}
\mathcal{S}_{L/R} \, \mathcal{G}_{ab} &= e^{-\varphi} \, \Big(\pm \overline{\lambda}_{L/R} \, \rho^{R/L}_{ab} - \frac{3}{2} \, i \, f_{[a} \, \overline{\chi}_{L/R} \, \Gamma_{b]} \, \lambda_{L/R} + \frac{3}{2} \, i \, \mathcal{D}_{[a} \, \overline{\chi}_{L/R} \, \Gamma_{b]} \, \lambda_{L/R}\\
&\qquad + \frac{3}{2} \, i \, \overline{\chi}_{L/R} \, \Gamma_{[a} \, \mathcal{L}^{(even)}_{b]\pm} \lambda_{L/R} + \frac{3}{2} \, i \, \overline{\chi}_{R/L} \, \Gamma_{[a} \, \mathcal{L}^{(odd)}_{b]\pm} \lambda_{L/R} \Big)\\
\mathcal{S}_{L/R} \, \mathcal{G}_{abcd} &= e^{-\varphi} \, \Big(\overline{\lambda}_{L/R} \, \Gamma_{[ab} \rho^{R/L}_{cd]} \pm \frac{i}{2} \, f_{[a} \, \overline{\chi}_{L/R} \, \Gamma_{bcd]} \, \lambda_{L/R} \mp \frac{i}{2} \, \mathcal{D}_{[a} \, \overline{\chi}_{L/R} \, \Gamma_{bcd]} \, \lambda_{L/R}\\
&\qquad \mp \frac{i}{2} \, \overline{\chi}_{L/R} \, \Gamma_{[abc} \, \mathcal{L}^{(even)}_{d]\pm} \lambda_{L/R} \pm \frac{i}{2} \, \overline{\chi}_{R/L} \, \Gamma_{[abc} \, \mathcal{L}^{(odd)}_{d]\pm} \lambda_{L/R}\\
&\qquad - \frac{3}{2} \, i \, \mathcal{H}_{[abc} \, \overline{\chi}_{L/R} \, \Gamma_{d]} \, \lambda_{L/R} \Big)\\
\mathcal{S}_{L/R} \, \mathcal{H}_{abc} &= \mp 2 \, i \, \overline{\lambda}_{L/R} \, \Gamma_{[a} \, \rho^{L/R}_{bc]}\\
\mathcal{S}_{L/R} \, \mathcal{D}_a \chi_{L/R} &= -\frac{1}{4} \, (\overline{\lambda}_{L/R} \, \Theta_{cd,a|R/L}) \, \Gamma^{cd} \, \chi_{L/R} + \Big[\mathcal{D}_a \mathcal{N}^{(even)}_{\pm} - (\mathcal{N} \, \mathcal{L}_a)^{(even)}_{\pm} \Big] \, \lambda_{L/R}\\
\mathcal{S}_{L/R} \, \mathcal{D}_a \chi_{R/L} &= -\frac{1}{4} \, (\overline{\lambda}_{L/R} \, \Theta_{cd,a|R/L}) \, \Gamma^{cd} \, \chi_{R/L} + \Big[\mathcal{D}_a \mathcal{N}^{(odd)}_{\pm} - (\mathcal{N} \, \mathcal{L}_a)^{(odd)}_{\pm} \Big] \, \lambda_{L/R}
\end{aligned}
$$

[1] In [26] will be shown that one can obtain a solution of the constraints (4.7)–(4.10) with 22 degrees of freedom, in a G_2 and in a SO(8) covariant basis. Finally, it is proven that the constraints are equivalent to Berkovits' constraints. As a side result, it is shown that also the geometrically-deduced constraints for IIA and IIB superstrings are consistent and equivalent.

[2] The pure spinor constraints for heterotic strings are derived from superembedding formalism in [29, 28].

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$$\begin{aligned}
\mathcal{S}_{L/R}\,\rho_{ab}^{L/R} &= \Upsilon^{(even)}_{ab\,\pm}\,\lambda_{L/R} - \frac{1}{4}\,R_{cd,ab}\,\Gamma^{ab}\,\lambda_{L/R} + 2\,\mathcal{P}_{L/R}[\lambda_{L/R}]\,\rho_{ab}^{L/R} \\
\mathcal{S}_{L/R}\,\rho_{ab}^{R/L} &= \Upsilon^{(odd)}_{ab\,\pm}\,\lambda_{L/R}
\end{aligned} \tag{4.12}$$

where we have used the following definitions

$$\begin{aligned}
(\mathcal{N}\,\mathcal{L}_a)^{(odd)}_{\pm} &\equiv \mathcal{N}^{(even)}_{\mp}\,\mathcal{L}^{(odd)}_{a\pm} + \mathcal{N}^{(odd)}_{\pm}\,\mathcal{L}^{(even)}_{a\pm} \\
(\mathcal{N}\,\mathcal{L}_a)^{(even)}_{\pm} &\equiv \mathcal{N}^{(even)}_{\pm}\,\mathcal{L}^{(even)}_{a\pm} + \mathcal{N}^{(odd)}_{\mp}\,\mathcal{L}^{(odd)}_{a\pm} \\
\Upsilon^{(even)}_{ab\,\pm} &= \mathcal{D}_{[a}\,\mathcal{L}^{(even)}_{b]\,\pm} + \mathcal{L}^{(even)}_{[a\pm}\,\mathcal{L}^{(even)}_{b]\pm} + \mathcal{L}^{(odd)}_{[a\mp}\,\mathcal{L}^{(odd)}_{b]\pm} \\
\Upsilon^{(odd)}_{ab\,\pm} &= \mathcal{D}_{[a}\,\mathcal{L}^{(odd)}_{b]\,\pm} + \mathcal{L}^{(odd)}_{[a\pm}\,\mathcal{L}^{(even)}_{b]\pm} + \mathcal{L}^{(even)}_{[a\mp}\,\mathcal{L}^{(odd)}_{b]\pm} \\
\mathcal{P}_{L/R}[\lambda_{L/R}] &= \pm\frac{21}{32}\,\Gamma_a\,\chi_{R/L}\,\overline{\lambda}_{L/R}\,\Gamma^a \mp \frac{1}{2560}\,\Gamma_{abcde}\,\chi_{R/L}\,\overline{\lambda}_{L/R}\,\Gamma^{abcde}\,.
\end{aligned}$$

We have concluded the derivation of the constrained BRST algebra for type IIA superstrings. Let us now go to step (*b*).

4.3 The antighosts and the Lagrange multipliers

The structure of the antighosts and of the Lagrange multipliers is motivated by the sort of gauging fixing one chooses to implement on the fermionic symmetries. Let us recall that in flat superspace the gravitino 1-form is the exterior derivative of the θ coordinates:

$$\psi_{L/R} = d\theta_{L/R} \quad \text{(flat superspace)} \tag{4.13}$$

and supersymmetry is nothing else but a translation in $\theta_{L/R}$:

$$\theta_{L/R} \mapsto \theta_{L/R} + \epsilon_{L/R}\,. \tag{4.14}$$

If we choose, as gauge fixing, the conditions:

$$\psi_R \wedge e^+ = 0 \quad ; \quad \psi_L \wedge e^- = 0 \tag{4.15}$$

we obtain that the spinor field θ_R is holomorphic on the world sheet while the spinor field θ_L is antiholomorphic on it. This is a very good starting point to obtain a two-dimensional conformal field theory from the pure spinor action we intend to construct. So, relying on this intuition based on the case of flat superspace, eq. (4.15) is singled out as our choice. There are no other compelling a-priori reasons to make such a choice but, once it is made, all the other steps are essentially determined and lead to an algorithmic derivation of the action.

Indeed, in order to obtain eqs. (4.15) as variational equations associated with Lagrange multiplier fields, we decide that these latter are a pair formed by a left handed $\mathrm{SO}(1,9)$ spinor $\mathbf{d}_+$ and a right handed $\mathrm{SO}(1,9)$ spinor $\mathbf{d}_-$ which will finally appear in the Lagrangian through terms of the following form:

$$\ldots + \overline{\mathbf{d}}_+\,\psi_R \wedge e^+ + \overline{\mathbf{d}}_-\,\psi_L \wedge e^- + \ldots \tag{4.16}$$

This choice determines also the representation assignments of the antighost fields $w_\pm$ which are introduced as those chiral spinors of ghost number $g = -1$ that play the role of predecessors of the $\mathbf{d}_\pm$ fields through the following relations:

$$
\begin{aligned}
\mathcal{S}_R \, w_+ &= \mathbf{d}_+ \\
\mathcal{S}_L \, w_+ &= 0 \\
\mathcal{S}_R \, w_- &= 0 \\
\mathcal{S}_L \, w_- &= \mathbf{d}_- \, .
\end{aligned}
\tag{4.17}
$$

From eq. (4.17) one might conclude that the BRST operators on the fields $\mathbf{d}_\pm$ necessarily make zero, but, as already anticipated, this is not the case. Indeed we can set:

$$
\begin{aligned}
\mathcal{S}_R \, \mathbf{d}_+ &= \xi_+ \\
\mathcal{S}_L \, \mathbf{d}_- &= \xi_- \\
\mathcal{S}_{L/R} \, \mathbf{d}_\pm &= 0
\end{aligned}
\tag{4.18}
$$

where $\xi_\pm$ encode a new gauge transformation that will be determined later.

Let us clarify this point. In [2], the pure spinor constraints are very simple since they do not interfere with the background, therefore it is straightforward to derive the gauge transformations for the conjugate momenta (notice that in [2] the Hamiltonian formalism has been used). In our case, the pure spinor constraints (4.7)–(4.10) involve the vielbein V^a and therefore one can wonder what is the interplay with the rest of the action to derive the correct gauge transformations. However, one can use the 1.5 formalism [27] and consider V^a as a non-dynamical field, then one derives the gauge transformations and, at the end, imposes the equations of motions by replacing V^a with the pullbacks $\Pi^a_\pm$. In [26], it is shown that, by using an adapted basis for the pullbacks, the amount of gauge symmetry is the correct one to give 22 degrees of freedom for the conjugate momenta $w_\pm$.

4.4 The BRST invariant type IIA superstring action

In [6] we constructed a BRST invariant action for the M2 brane with pure spinors where we used a certain gauge fixing term. This construction seems incomplete because it was based on a solution of the pure spinor constraints which was not complete. There was an idea that the gauge fixing term could be related to the cohomology class which defines the FDA but also this idea appears now doubtful. Indeed the M2 brane action we constructed has no term of the type:

$$
\overline{\mathbf{d}} \, \Gamma_{abcd} \cdot \mathcal{F}^{abcd} \, \mathbf{d}
\tag{4.19}
$$

where $\mathbf{d}$ is the Lagrange multiplier field and $\mathcal{F}_{abcd}$ denotes the 4-index field strength. This is a clear indication that the assumptions made were too restrictive since a term of the form (4.19) is the vertex of Ramond Ramond fields and it is an essential part of the Berkovits' superstring Lagrangian. On the other hand this latter should be related to the M2-brane action by double dimensional reduction, at least in the case of type IIA theory and this cannot produce terms of the form (4.19) if they are missing in higher dimension.

Hence it is mandatory to repeat the construction of the type IIA and also of the type IIB pure spinor superstrings from scratch. Differently from what it was assumed by Berkovits we have shown that the constraints are not the same in all cases and, in particular, they are not the same for type IIB and type IIA superstrings. Moreover they feel the background and are not given once for all. In a separate forthcoming publication [26] two of us will discuss the solution of the new formulation of pure spinor constraints streaming from FDA and rheonomy. Anticipating the result proved in [26], we state that, notwithstanding their different structure the background dependent constraints derived from the FDA lead to the same counting of degrees of freedom as in Berkovits' approach both in the type IIA and type IIB case, namely 22. In the case of type IIA, which is presently under consideration the pure spinor constraints are given by eqs. (4.7), (4.8), (4.9), (4.10). In the presence of these constraints and using the Lagrange multiplier and antighosts discussed in the previous section we now construct an addendum $\mathcal{A}_{\mathrm{gf}}^{\mathrm{IIA}}$ to the Green-Schwarz action such that its BRST variation exactly cancels the BRST variation of the latter:

$$(\mathcal{S}_L + \mathcal{S}_R)\, \mathcal{A}_{\mathrm{gf}}^{\mathrm{IIA}} = -\,(\mathcal{S}_L + \mathcal{S}_R)\, \mathcal{A}_{\mathrm{GS}}\,. \tag{4.20}$$

In order to perform such a construction we begin by writing down the BRST variation of the Green-Schwarz action. This is immediately obtained from eq. (3.11) by replacing the supersymmetry parameter with the pure spinor superghost and setting the parameter q to its value $q = 1$:

$$(\mathcal{S}_L + \mathcal{S}_R)\, \mathcal{A}_{\mathrm{GS}} = \int 2\,\mathrm{i}\,\left[\overline{\lambda_R}\,\mathbf{\Gamma}_+\psi_R \wedge e^+ - \overline{\lambda_L}\,\mathbf{\Gamma}_-\psi_L \wedge e^-\right]. \tag{4.21}$$

Next we introduce the following ansatz for the *gauge fixing* action:

$$\begin{aligned}\mathcal{A}_{\mathrm{gf}}^{\mathrm{IIA}} &= \mathcal{S}_R\left(\overline{w}_+\,\psi_R \wedge e^+\right) + \mathcal{S}_L\left(\overline{w}_-\,\psi_L \wedge e^-\right)\\ &\quad + \mathcal{S}_R\,\mathcal{S}_L\left(\overline{w}_+\,\Omega\, w_-\, e^+ \wedge e^-\right)\end{aligned} \tag{4.22}$$

where Ω is a matrix in spinor space constructed by saturating gamma matrices only with physical curvature components. The precise form of Ω will now be determined by imposing eq. (4.20). The fact that it depends on physical fields only implies that the action of the operators $\mathcal{S}_L^2$ and $\mathcal{S}_R^2$ on them are zero modulo Lorentz transformations and this allows the following formal manipulations:

$$\begin{aligned}(\mathcal{S}_L + \mathcal{S}_R)\, A_{\mathrm{gf}}^{\mathrm{IIA}} &= \mathcal{S}_R^2\left(\overline{w}_+\,\psi_R \wedge e^+\right) + \mathcal{S}_L^2\left(\overline{w}_-\,\psi_L \wedge e^-\right)\\ &\quad - \mathcal{S}_R\,\mathcal{S}_L\left(\overline{w}_+\,\psi_R \wedge e^+\right) - \mathcal{S}_L\,\mathcal{S}_R\left(\overline{w}_+\,\psi_R \wedge e^+\right)\\ &\quad + \mathcal{S}_L\,\mathcal{S}_R^2\left(\overline{w}_+\,\Omega\, w_-\, e^+ \wedge e^-\right)\\ &\quad - \mathcal{S}_R\,\mathcal{S}_L^2\left(\overline{w}_+\,\Omega\, w_-\, e^+ \wedge e^-\right).\end{aligned} \tag{4.23}$$

Next taking into account that the only field on which $\mathcal{S}_{L/R}^2$ is non zero is $w_{\mp}$, from eq. (4.23) we obtain:

$$\begin{aligned}(\mathcal{S}_L + \mathcal{S}_R)\, A_{\mathrm{gf}}^{\mathrm{IIA}} &= \overline{\xi}_+\,\psi_R \wedge e^+ + \xi_-\psi_L \wedge e^-\\ &\quad + \mathcal{S}_R\left[\overline{w}_+\,\mathcal{S}_L\,(\psi_R) \wedge e^+ - \overline{w}_+\,\Omega\,\xi_-\, e^+ \wedge e^-\right]\\ &\quad + \mathcal{S}_L\left[\overline{w}_-\,\mathcal{S}_R\,(\psi_L) \wedge e^- + \overline{\xi}_+\,\Omega\, w_-\, e^+ \wedge e^-\right].\end{aligned} \tag{4.24}$$

Combining these results with the BRST variation of the Green-Schwarz action given in eq. (4.21) conclude that we have BRST invariance of the complete action, namely:

$$(\mathcal{S}_L + \mathcal{S}_R)\left(\mathcal{A}_{\rm GS} + \mathcal{A}^{\rm IIA}_{\rm gf}\right) = 0 \tag{4.25}$$

if the following conditions are verified:

$$\overline{\xi}_+ \psi_R \wedge e^+ = -2\,\mathrm{i}\overline{\lambda}_R \mathbf{\Gamma}_+ \psi_R \wedge e^+ , \tag{4.26}$$
$$\overline{\xi}_- \psi_L \wedge e^- = 2\,\mathrm{i}\overline{\lambda}_L \mathbf{\Gamma}_- \psi_L \wedge e^- , \tag{4.27}$$

and moreover if the arguments of $\mathcal{S}_{R/L}$ in the last two lines of eq. (4.24) vanish separately. Conditions (4.26), (4.27) allow to determine the gauge transformation of the anti-ghost fields, namely $\xi_\pm$. We indeed find:

$$\bar{\xi}_+ = -2i\,\bar{\lambda}_R \mathbf{\Gamma}_+ \; ; \; \bar{\xi}_- = 2i\,\bar{\lambda}_L \mathbf{\Gamma}_- , \tag{4.28}$$

or, equivalently,

$$\xi_+ = 2i\,\mathbf{\Gamma}_+ \lambda_R \; ; \; \xi_- = -2i\,\mathbf{\Gamma}_- \lambda_L . \tag{4.29}$$

We next require the vanishing of the arguments of $\mathcal{S}_{R/L}$ in the last two lines of eq. (4.24). This implies

$$\begin{aligned} 0 &= \overline{w}_+ \mathcal{S}_L\,(\psi_R) \wedge e^+ - \overline{w}_+ \Omega\, \xi_-\, e^+ \wedge e^- = \\ &= \overline{w}_+ \mathcal{M}_- \mathbf{\Gamma}_- \lambda_L\, e^+ \wedge e^- - 2\,i\,\overline{w}_+ \Omega\, \mathbf{\Gamma}_- \lambda_L\, e^+ \wedge e^- , \end{aligned} \tag{4.30}$$

where we have used the last of equations (4.5) to express $\mathcal{S}_{L/R}\left(\psi_{R/L}\right)$. Equation (4.30) is satisfied provided we make the following identification:

$$\mathcal{M}_- = 2\,i\,\Omega . \tag{4.31}$$

The second condition reads:

$$\begin{aligned} 0 &= \overline{w}_- \mathcal{S}_R\,(\psi_L) \wedge e^- + \overline{\xi}_+ \Omega\, w_-\, e^+ \wedge e^- = \\ &= \overline{w}_- \mathcal{M}_+ \mathbf{\Gamma}_+ \lambda_R\, e^+ \wedge e^- - 2\,i\,\bar{\lambda}_R \mathbf{\Gamma}_+ \Omega\, w_-\, e^+ \wedge e^- . \end{aligned} \tag{4.32}$$

Now we may use the following property:

$$\overline{w}_- \mathcal{M}_+ \mathbf{\Gamma}_+ \lambda_R = w_-^T C\, \mathcal{M}_+ \mathbf{\Gamma}_+ \lambda_R = \lambda_R^T C \mathbf{\Gamma}_+ C^{-1} \mathcal{M}_+^T C w_- = \bar{\lambda}_R \mathbf{\Gamma}_+ \widetilde{\mathcal{M}_+} w_- , \tag{4.33}$$

where C denotes the charge conjugation matrix, defined by the property $C^{-1}\,\Gamma_a\, C = -\Gamma_a^T$, and $\widetilde{\mathcal{M}_\pm} = C^{-1}\,\mathcal{M}_\pm^T\, C$. Equation (4.32) then implies

$$\widetilde{\mathcal{M}_+} = 2\,i\,\Omega , \tag{4.34}$$

which is consistent with (4.31) since

$$\widetilde{\mathcal{M}_\pm} = \mathcal{M}_\mp . \tag{4.35}$$

4.5 Explicit form of the pure spinor σ-model action

Here we explicitly compute the terms coming form new piece of the action denoted by $\mathcal{A}_{\rm gf}^{IIA}$ by acting with the BRST operators $\mathcal{S}_L$ and $\mathcal{S}_R$ on the "gauge-fixing" terms

$$\begin{aligned}\mathcal{A}_{\rm gf}^{\rm IIA} &= \mathcal{S}_R\left(\overline{w}_+\,\psi_R\wedge e^+\right)+\mathcal{S}_L\left(\overline{w}_-\,\psi_L\wedge e^-\right)\\ &\quad+\mathcal{S}_R\,\mathcal{S}_L\left(\overline{w}_+\,\Omega\, w_-\,e^+\wedge e^-\right)\\ &=\overline{\mathbf{d}}_+\,\psi_R\wedge e^+ +\overline{\mathbf{d}}_-\,\psi_L\wedge e^- +\frac{\mathrm{i}}{2}\overline{\mathbf{d}}_+\,\mathcal{M}_-\,\mathbf{d}_-\\ &\quad-\overline{w}_+\left(\mathcal{S}_R\psi_R\right)\wedge e^+ -\overline{w}_-\left(\mathcal{S}_L\psi_L\right)\wedge e^-\\ &\quad-\frac{\mathrm{i}}{2}\,\overline{w}_+\left(\mathcal{S}_R\mathcal{M}_-\right)\mathbf{d}_- +\frac{\mathrm{i}}{2}\,\overline{\mathbf{d}}_+\left(\mathcal{S}_L\mathcal{M}_-\right)w_- -\frac{\mathrm{i}}{2}\,\overline{w}_+\left(\mathcal{S}_R\mathcal{S}_L\mathcal{M}_-\right)w_-\end{aligned}\tag{4.36}$$

where the action of $\mathcal{S}$ on $\psi_{L/R}$ is given in (4.5), while the action of $\mathcal{S}$ on the spinor matrices $\mathcal{M}_\pm$ can be deduced by computing the corresponding BRTS variation of the tensors in (2.20), which read as follows

$$\begin{aligned}\mathcal{S}_{L/R}\mathcal{M}_- &= \pm\frac{i}{8}\,\overline{\lambda}_{L/R}\,\rho_{ab}^{R/L}\,\Gamma^{ab}-\frac{i}{16}\,\overline{\lambda}_{L/R}\,\Gamma_{ab}\rho_{cd}^{R/L}\,\Gamma^{abcd}-\frac{3}{16}\,\overline{\lambda}_{L/R}\,\Gamma_{[a}\,\mathcal{D}_{b]}\chi_{L/R}\,\Gamma^{ab}\\ &\quad\pm\frac{1}{32}\,\overline{\lambda}_{L/R}\,\Gamma_{[abc}\,\mathcal{D}_{d]}\chi_{L/R}\,\Gamma^{ab}+\overline{\chi}_R\,\mathcal{A}^-_{L\,|\lambda_{R/L}=0}+\overline{\chi}_L\,\mathcal{A}^-_{R\,|\lambda_{R/L}=0}\end{aligned}\tag{4.37}$$

where we have defined $\mathcal{A}^-_{L/R}$ in the following way

$$\begin{aligned}\mathcal{A}^-_{L/R} &= \Bigg(\pm\frac{3}{16}\,i\,\lambda_{L/R}\,e^{\varphi}\,G_{ab}+\frac{3}{16}\,f_{[a}\,\Gamma_{b]}\,\lambda_{R/L}-\frac{3}{16}\,\Gamma_{[a}\mathcal{L}^{(even)}_{b]\mp}\,\lambda_{R/L}-\frac{3}{16}\,\Gamma_{[a}\mathcal{L}^{(odd)}_{b]\pm}\,\lambda_{L/R}\\ &\qquad\pm\frac{9}{64}\,i\,\Gamma_{ab}\mathcal{N}^{(even)}_{\pm}\,\lambda_{L/R}\pm\frac{9}{64}\,i\,\Gamma_{ab}\mathcal{N}^{(odd)}_{\mp}\,\lambda_{R/L}\Bigg)\otimes\Gamma^{ab}\\ &\quad+\Bigg(\mp\frac{3}{32}\,i\,\lambda_{L/R}\,e^{\varphi}\,G_{abcd}-\frac{3}{32}\,\mathcal{H}_{[abc}\,\Gamma_{d]}\,\lambda_{R/L}\pm\frac{1}{32}\,\Gamma_{[abc}\mathcal{L}^{(even)}_{d]\mp}\,\lambda_{R/L}\\ &\qquad\pm\frac{1}{32}\,\Gamma_{[abc}\mathcal{L}^{(odd)}_{d]\pm}\,\lambda_{L/R}\mp\frac{1}{32}\,f_{[a}\,\Gamma_{bcd]}\,\lambda_{R/L}-\frac{3\,i}{256}\,\Gamma_{abcd}\mathcal{N}^{(even)}_{\pm}\,\lambda_{L/R}\\ &\qquad-\frac{3\,i}{256}\,\Gamma_{abcd}\mathcal{N}^{(odd)}_{\mp}\,\lambda_{R/L}\Bigg)\otimes\Gamma^{abcd}.\end{aligned}\tag{4.38}$$

The complete expression of $\mathcal{S}_R\,\mathcal{S}_L\,\mathcal{M}_-$, which can be computed using the above formulas, is rather involved. Therefore we shall give it below for $\chi=0$

$$\begin{aligned}\left[\mathcal{S}_R\,\mathcal{S}_L\,\mathcal{M}_-\right]_{\chi=0} &= -\frac{i}{8}\,\overline{\lambda}_L\left(\Upsilon^{(even)}_{ab-}-\frac{1}{4}\,R_{cd,ab}\,\Gamma^{cd}\right)\lambda_R\,\Gamma^{ab}\\ &\quad+\frac{i}{16}\,\overline{\lambda}_L\,\Gamma_{ab}\left(\Upsilon^{(even)}_{cd-}-\frac{1}{4}\,R_{ef,cd}\,\Gamma^{ef}\right)\lambda_R\,\Gamma^{abcd}\\ &\quad+\frac{3}{16}\,\overline{\lambda}_L\,\Gamma_a\left(\mathcal{D}_b\mathcal{N}^{(odd)}_- -\left(\mathcal{N}\,\mathcal{L}_b\right)^{(odd)}_-\right)\lambda_R\,\Gamma^{ab}\\ &\quad-\frac{1}{32}\,\overline{\lambda}_L\,\Gamma_{abc}\left(\mathcal{D}_d\mathcal{N}^{(odd)}_- -\left(\mathcal{N}\,\mathcal{L}_d\right)^{(odd)}_-\right)\lambda_R\,\Gamma^{abcd}+\overline{\lambda}_R\widetilde{\mathcal{N}^{(even)}_-}\,\mathcal{A}^-_{L\,|\lambda_R=0}\\ &\quad+\overline{\lambda}_R\widetilde{\mathcal{N}^{(odd)}_-}\,\mathcal{A}^-_{R\,|\lambda_R=0}\end{aligned}\tag{4.39}$$

5. Conclusions

We provided a complete geometrical derivation of the pure spinor sigma model for type IIA superstrings based on the FDA of the corresponding supergravity. The FDA formulation of the latter had to be adapted to this problem by using directly the string frame rather than the Einstein frame. This require a field redefinition. It turned out that the solution of the Bianchi identities and the construction of the supergravity rheonomic parametrization was much easier derived directly in the string frame than obtainined it from field redefinitions and dimensional reduction starting from 11d. From this effort, we gained a very simple rule for the BRST transformations to be used for the pure spinor formulation. The latter is obtained in a Lagrangian formalism and the result has the advantage to relate the superfields appearing in the FDA with those appearing in the BRST transformation and in the sigma model. That is important to have in order a straight path for constructing the sigma model given in any supergravity background.

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A. Summary tables

B. Derivation of type IIA supergravity in the string frame

The derivation of type IIA supergravity was done in two steps. In the first one, we started from the $D = 11$ supergravity FDA and from its rheonomic parametrization and we reduce them on a circle. Next, we perform a Weyl rescaling and gravitino field redefinition to go to the string frame. In the second step, we derived the rheonomic parametrization directly by solving the Bianchi identities in the $D = 10$ in the string frame. Here, we just sketch such a derivation.

B.1 The D=11 FDA

We start from the FDA of M-theory whose complete set of curvatures is given below [17, 18]:

$$\mathbf{T}^{\overline{a}} = \mathcal{D}\mathbf{V}^{\overline{a}} - \mathrm{i}\frac{1}{2}\,\overline{\Psi} \wedge \Gamma^{\overline{a}}\,\Psi \tag{B.1}$$

$$\mathbf{R}^{\overline{ab}} = d\mathbf{\Omega}^{\overline{ab}} - \Omega^{\overline{ac}} \wedge \Omega^{\overline{cb}} \tag{B.2}$$

$$\widehat{\rho} = \mathcal{D}\Psi \equiv d\Psi - \frac{1}{4}\,\Omega^{\overline{ab}} \wedge \Gamma_{\overline{ab}}\,\Psi \tag{B.3}$$

$$\mathbb{F}^{[4]} = d\mathbb{A}^{[3]} - \frac{1}{2}\,\overline{\Psi} \wedge \Gamma_{\overline{ab}}\,\Psi \wedge \mathbf{V}^{\overline{a}} \wedge \mathbf{V}^{\overline{b}} \tag{B.4}$$

$$\begin{aligned}\mathbb{F}^{[7]} &= d\mathbb{A}^{[6]} - 15\,\mathbb{F}^{[4]} \wedge \mathbb{A}^{[3]} - \frac{15}{2}\,\mathbf{V}^{\overline{a}} \wedge \mathbf{V}^{\overline{b}} \wedge \bar{\Psi} \wedge \Gamma_{\overline{ab}}\,\Psi \wedge \mathbb{A}^{[3]} \\ &\quad -\mathrm{i}\,\frac{1}{2}\,\overline{\Psi} \wedge \Gamma_{\overline{a_1 \dots a_5}}\,\Psi \wedge \mathbf{V}^{\overline{a_1}} \wedge \dots \wedge \mathbf{V}^{\overline{a_5}}\end{aligned} \tag{B.5}$$

$$
\begin{array}{|rcl|}
\hline
\mathcal{M}_{ab} & = & \left(\frac{1}{8}\,\exp[\varphi]\,\mathcal{G}_{ab} \,+\, \frac{9}{64}\,\overline{\chi}_R\,\Gamma_{ab}\,\chi_L\right) \\
\mathcal{M}_{abcd} & = & -\,\frac{1}{16}\,\exp[\varphi]\,\mathcal{G}_{abcd} \,-\, \frac{3i}{256}\,\overline{\chi}_L\,\Gamma_{abcd}\,\chi_R \\
\mathcal{N}_0 & = & \frac{3}{4}\,\overline{\chi}_L\,\chi_R \\
\mathcal{N}_{ab} & = & \frac{1}{4}\,\exp[\varphi]\,\mathcal{G}_{ab} \,+\, \frac{9}{32}\,\overline{\chi}_R\,\Gamma_{ab}\,\chi_L = 2\,\mathcal{M}_{ab} \\
\mathcal{N}_{abcd} & = & \frac{1}{24}\,\exp[\varphi]\,\mathcal{G}_{abcd} + \frac{1}{128}\,\overline{\chi}_R\,\Gamma_{abcd}\,\chi_L = -\frac{2}{3}\mathcal{M}_{abcd} \\
\hline
\mathcal{Z} & = & \mathcal{N}_{ab}\Gamma^{ab} \,+\, 3\,\mathcal{N}_{abcd}\,\Gamma^{abcd} \\
\mathcal{M}_{\pm} & = & \mathrm{i}\,\left(\mp\mathcal{M}_{ab}\,\Gamma^{ab} \,+\, \mathcal{M}_{abcd}\,\Gamma^{abcd}\right) \\
\mathcal{N}^{(even)}_{\pm} & = & \mp\mathcal{N}_0\,\mathbf{1} \,+\, \mathcal{N}_{ab}\,\Gamma^{ab} \,\mp\, \mathcal{N}_{abcd}\,\Gamma^{abcd} \\
\mathcal{N}^{(odd)}_{\pm} & = & \pm\frac{i}{3}\,f_a\,\Gamma^a \pm \frac{1}{64}\,\overline{\chi}_{R/L}\,\Gamma_{abc}\,\chi_{R/L}\,\Gamma^{abc} - \frac{i}{12}\,\mathcal{H}_{abc}\,\Gamma^{abc} \\
\mathcal{L}^{(odd)}_{a\,\pm} & = & \mathcal{M}_{\mp}\,\Gamma_a\ ;\ \ \mathcal{L}^{(even)}_{a\,\pm} = \mp\frac{3}{8}\,\mathcal{H}_{abc}\,\Gamma^{bc} \\
\hline
(\mathcal{N}\,\mathcal{L}_a)^{(odd)}_{\pm} & \equiv & \mathcal{N}^{(even)}_{\mp}\,\mathcal{L}^{(odd)}_{a\pm} + \mathcal{N}^{(odd)}_{\pm}\,\mathcal{L}^{(even)}_{a\pm} \\
(\mathcal{N}\,\mathcal{L}_a)^{(even)}_{\pm} & \equiv & \mathcal{N}^{(even)}_{\pm}\,\mathcal{L}^{(even)}_{a\pm} + \mathcal{N}^{(odd)}_{\mp}\,\mathcal{L}^{(odd)}_{a\pm} \\
\Upsilon^{(even)}_{ab\,\pm} & = & \mathcal{D}_{[a}\,\mathcal{L}^{(even)}_{b]\,\pm} + \mathcal{L}^{(even)}_{[a\pm}\,\mathcal{L}^{(even)}_{b]\pm} + \mathcal{L}^{(odd)}_{[a\mp}\,\mathcal{L}^{(odd)}_{b]\pm} \\
\Upsilon^{(odd)}_{ab\,\pm} & = & \mathcal{D}_{[a}\,\mathcal{L}^{(odd)}_{b]\,\pm} + \mathcal{L}^{(odd)}_{[a\pm}\,\mathcal{L}^{(even)}_{b]\pm} + \mathcal{L}^{(even)}_{[a\mp}\,\mathcal{L}^{(odd)}_{b]\pm} \\
\mathcal{P}_{L/R}[\lambda_{L/R}] & = & \pm\frac{21}{32}\,\Gamma_a\,\chi_{R/L}\,\overline{\lambda}_{L/R}\,\Gamma^a \mp \frac{1}{2560}\,\Gamma_{abcde}\,\chi_{R/L}\,\overline{\lambda}_{L/R}\,\Gamma^{abcde} \\
\hline
\end{array}
$$

Table 3: Tensors and matrices: recalling that $\mathcal{G}_{ab}$ and $\mathcal{G}_{abcd}$ denote the supercovariant field strengths of the Ramond Ramond 1-form and 3-form respectively, $\mathcal{H}_{abc}$ the supercovariant field strength of the Neveu Schwarz two-form, while $\chi_{L/R}$ denote the chiral components of the dilatino spinor field and φ, f_a denote the dilaton and its supercovariant derivative, the table above summarizes the precise definition of certain tensors and matrices appearing both in the sigma model action and in the BRST transformation rules.

In the above equations $\mathbf{V}^{\overline{a}}$ and $\mathbf{\Omega}^{\overline{ab}}$ are respectively the $11D$ vielbein and spin connection, Ψ is the $11D$ gravitino, namely a Majorana spinor valued 1-form of fermionic type with 32-components, while $\mathbb{A}^{[3]}$ and $\mathbb{A}^{[6]}$ are a bosonic 3-form and a bosonic 6-form respectively. Equations (B.1), (B.2), (B.3) define the curvatures of the $11D$ superPoincaré algebra. According to Sullivan's second theorem the 3-form $\mathbb{A}^{[3]}$ corresponds to the first FDA extension of this latter generated by a degree 4 cohomology class, while the 6-form $\mathbb{A}^{[6]}$ corresponds to a further extension of the FDA generated by a degree 7 cohomology class of the first extension.

The rheonomic parametrization of the M-theory curvatures is the following one:

$$\mathbf{T}^{\overline{a}} = 0 \tag{B.6}$$

$$\mathbb{F}^{[4]} = F_{\overline{a_1\dots a_4}}\,\mathbf{V}^{\overline{a_1}} \wedge \cdots \wedge \mathbf{V}^{\overline{a_4}} \tag{B.7}$$

$$\mathbb{F}^{[7]} = \frac{1}{84}F^{\overline{a_1\dots a_4}}\,\mathbf{V}^{\overline{b_1}} \wedge \cdots \wedge \mathbf{V}^{\overline{b_7}}\,\epsilon_{\overline{a_1\dots a_4 b_1\dots b_7}} \tag{B.8}$$

$$\widehat{\rho} = \rho_{\overline{a_1 a_2}}\mathbf{V}^{\overline{a_1}} \wedge \mathbf{V}^{\overline{a_2}} + \mathrm{i}\frac{1}{3}\left(\Gamma^{\overline{a_1 a_2 a_3}}\Psi \wedge \mathbf{V}^{\overline{a_4}} - \frac{1}{8}\Gamma^{\overline{a_1\dots a_4 m}}\Psi \wedge \mathbf{V}^{\overline{m}}\right) F_{\overline{a_1\dots a_4}} \tag{B.9}$$

$$
\begin{array}{|rl|}
\hline
\mathcal{S}_{L/R}\,\mathbf{B}^{[2]} &= \mp\, 2\,\mathrm{i}\,\overline{\psi}_{L/R}\,\Gamma_a\,\lambda_{L/R}\,V^a \\
\mathcal{S}_{L/R}\,\mathbf{C}^{[1]} &= \mp\,\exp[-\varphi]\,\overline{\psi}_{R/L}\,\lambda_{L/R} + \frac{3}{2}\,\mathrm{i}\,\exp[-\varphi]\,\overline{\chi}_{L/R}\,\Gamma_a\,\lambda_{L/R}\,V^a \\
\mathcal{S}_{L/R}\,\mathbf{C}^{[3]} &= \overline{\psi}_{R/L}\,\Gamma_{ab}\,\lambda_{L/R}\,V^a \wedge V^b - B^{[2]} \wedge \mathcal{S}_{L/R}C^{[1]} \\
&\quad \mp\,\mathrm{i}\,\frac{1}{2}\,\exp[-\varphi]\,\overline{\chi}_{L/R}\,\Gamma_{abc}\,\lambda_{L/R}\,V^a \wedge V^b \wedge V^c \\
\mathcal{S}_{L/R}\,V^a &= \mathrm{i}\,\overline{\psi}_{L/R}\,\Gamma^a\,\lambda_{L/R} \\
\mathcal{S}_{L/R}\psi_{L/R} &= -\mathcal{D}\,\lambda_{L/R} \mp \frac{3}{8}\,\Gamma^{a_1a_2}\,\lambda_{L/R}\,V^{a_3}\,\mathcal{H}_{a_1a_2a_3} \pm \frac{21}{16}\,\Gamma_a\chi_{R/L}\,(\overline{\psi}_{L/R}\,\Gamma^a\lambda_{L/R}) \\
&\quad \mp\frac{1}{1280}\,\Gamma_{a_1\dots a_5}\chi_{R/L}\,(\overline{\psi}_{L/R}\,\Gamma^{a_1\dots a_5}\lambda_{L/R}) \\
\mathcal{S}_{R/L}\psi_{L/R} &= \mathcal{M}_{\pm}\,\Gamma_b\lambda_{R/L}\,V^b \\
\hline
\mathcal{S}_{L/R}\lambda_{L/R} &= \pm\frac{21}{16}\,\Gamma_a\chi_{R/L}\,(\overline{\lambda}_{L/R}\,\Gamma^a\lambda_{L/R}) \\
&\quad \mp\frac{1}{1280}\,\Gamma_{a_1\dots a_5}\chi_{R/L}\,(\overline{\lambda}_{L/R}\,\Gamma^{a_1\dots a_5}\lambda_{L/R} \\
\mathcal{S}_{R/L}\lambda_{L/R} &= 0 \\
\hline
\mathcal{S}_R\,w_+ &= \mathbf{d}_+ \\
\mathcal{S}_L\,w_+ &= 0 \\
\mathcal{S}_R\,w_- &= 0 \\
\mathcal{S}_L\,w_- &= \mathbf{d}_- \\
\mathcal{S}_R\,\mathbf{d}_+ &= 2\mathrm{i}\,\boldsymbol{\Gamma}_a\,\Pi^a_+\,\lambda_R \\
\mathcal{S}_L\,\mathbf{d}_- &= -2\mathrm{i}\,\boldsymbol{\Gamma}_a\,\Pi^a_-\,\lambda_L \\
\mathcal{S}_{L/R}\,\mathbf{d}_{\pm} &= 0 \\
\hline
\end{array}
$$

Table 4: BRST algebra: in this table we summarize the BRST transformations of the fundamental fields. In the first box are displayed the BRST transformations of the physical fields encoded in the supergravity forms: the vielbein V^a, the NS 2-form $\mathbf{B}^{[2]}$, the Ramond Ramond forms $\mathbb{C}^{[1,3]}$ and the gravitino $\psi_{L/R}$. In the second box those of the (pure spinor) superghosts $\lambda_{L/R}$, while the third box gives the transformations of the antighosts $w_{\pm}$ and of the Lagrange multipliers $\mathbf{d}_{\pm}$.

$$
\begin{array}{|rl|}
\hline
\mathcal{S}_{L/R}\,\mathcal{G}_{ab} &= e^{-\varphi}\,\Big(\pm\overline{\lambda}_{L/R}\,\rho^{R/L}_{ab} - \frac{3}{2}\,i\,f_{[a}\,\overline{\chi}_{L/R}\,\Gamma_{b]}\,\lambda_{L/R} + \frac{3}{2}\,i\,\mathcal{D}_{[a}\,\overline{\chi}_{L/R}\,\Gamma_{b]}\,\lambda_{L/R} \\
&\quad +\frac{3}{2}\,i\,\overline{\chi}_{L/R}\,\Gamma_{[a}\,\mathcal{L}^{(even)}_{b]\pm}\lambda_{L/R} + \frac{3}{2}\,i\,\overline{\chi}_{R/L}\,\Gamma_{[a}\,\mathcal{L}^{(odd)}_{b]\pm}\lambda_{L/R}\Big) \\
\mathcal{S}_{L/R}\,\mathcal{G}_{abcd} &= e^{-\varphi}\,\Big(\overline{\lambda}_{L/R}\,\Gamma_{[ab}\rho^{R/L}_{cd]} \pm \frac{i}{2}\,f_{[a}\,\overline{\chi}_{L/R}\,\Gamma_{bcd]}\,\lambda_{L/R} \mp \frac{i}{2}\,\mathcal{D}_{[a}\,\overline{\chi}_{L/R}\,\Gamma_{bcd]}\,\lambda_{L/R} \\
&\quad \mp\frac{i}{2}\,\overline{\chi}_{L/R}\,\Gamma_{[abc}\,\mathcal{L}^{(even)}_{d]\pm}\lambda_{L/R} \pm \frac{i}{2}\,\overline{\chi}_{R/L}\,\Gamma_{[abc}\,\mathcal{L}^{(odd)}_{d]\pm}\lambda_{L/R} - \frac{3}{2}\,i\,\mathcal{H}_{[abc}\,\overline{\chi}_{L/R}\,\Gamma_{d]}\,\lambda_{L/R}\Big) \\
\mathcal{S}_{L/R}\,\mathcal{H}_{abc} &= \mp 2\,i\,\overline{\lambda}_{L/R}\,\Gamma_{[a}\,\rho^{L/R}_{bc]} \\
\mathcal{S}_{L/R}\,\mathcal{D}_a\chi_{L/R} &= -\frac{1}{4}\,(\overline{\lambda}_{L/R}\,\Theta_{cd,a|R/L})\,\Gamma^{cd}\,\chi_{L/R} + \Big[\mathcal{D}_a\mathcal{N}^{(even)}_{\pm} - (\mathcal{N}\,\mathcal{L}_a)^{(even)}_{\pm}\Big]\,\lambda_{L/R} \\
\mathcal{S}_{L/R}\,\mathcal{D}_a\chi_{R/L} &= -\frac{1}{4}\,(\overline{\lambda}_{L/R}\,\Theta_{cd,a|R/L})\,\Gamma^{cd}\,\chi_{R/L} + \Big[\mathcal{D}_a\mathcal{N}^{(odd)}_{\pm} - (\mathcal{N}\,\mathcal{L}_a)^{(odd)}_{\pm}\Big]\,\lambda_{L/R} \\
\mathcal{S}_{L/R}\,\rho^{L/R}_{ab} &= \Upsilon^{(even)}_{ab\,\pm}\,\lambda_{L/R} - \frac{1}{4}\,R_{cd,ab}\,\Gamma^{ab}\,\lambda_{L/R} + 2\,\mathcal{P}_{L/R}[\lambda_{L/R}]\,\rho^{L/R}_{ab} \\
\mathcal{S}_{L/R}\,\rho^{R/L}_{ab} &= \Upsilon^{(odd)}_{ab\,\pm}\,\lambda_{L/R} \\
\hline
\end{array}
$$

Table 5: BRST algebra: in this table we display the BRST transformations of the various field strenghts.

$$
\begin{aligned}
\mathbf{R}^{\overline{ab}} = {} & R^{\overline{ab}}{}_{\overline{cd}}\,\mathbf{V}^{\overline{c}} \wedge \mathbf{V}^{\overline{d}} + \mathrm{i}\,\overline{\rho}_{\overline{mn}}\left(\frac{1}{2}\Gamma^{\overline{abmnc}} - \frac{2}{9}\Gamma^{\overline{mn[a}}\,\delta^{\overline{b]c}} + 2\,\Gamma^{\overline{ab[m}}\,\delta^{\overline{n]c}}\right)\Psi \wedge \mathbf{V}_{\overline{c}} \\
& + \overline{\Psi} \wedge \Gamma_{\overline{mn}}\,\Psi\,F^{\overline{mnab}} + \frac{1}{24}\overline{\Psi} \wedge \Gamma^{\overline{abc_1\dots c_4}}\,\Psi\,F_{\overline{c_1\dots c_4}}
\end{aligned} \tag{B.10}
$$

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$$
\boxed{
\begin{aligned}
\mathcal{A} &= \mathcal{A}_{\rm GS} + \mathcal{A}^{\rm IIA}_{\rm gf} \\
\mathcal{A}_{\rm GS} &= \int \Big(\Pi^a_+ \, V^b \, \eta_{ab} \, \wedge \, e^+ \; - \; \Pi^a_- \, V^b \, \eta_{ab} \, \wedge \, e^- \; + \; \tfrac{1}{2} \Pi^a_i \, \Pi^b_j \, \eta^{ij} \, \eta_{ab} \, e^+ \, \wedge \, e^- \; + \tfrac{1}{2} \, \mathbf{B}^{[2]} \Big) \\
\mathcal{A}^{\rm IIA}_{\rm gf} &= \int \Big(\overline{\mathbf{d}}_+ \, \psi_R \, \wedge \, e^+ + \overline{\mathbf{d}}_- \, \psi_L \, \wedge \, e^- + \tfrac{\rm i}{2} \overline{\mathbf{d}}_+ \, \mathcal{M}_- \, \mathbf{d}_- \\
& \quad - \overline{w}_+ \, (\mathcal{S}_R \psi_R) \, \wedge \, e^+ - \overline{w}_- \, (\mathcal{S}_L \psi_L) \, \wedge \, e^- \\
& \quad - \tfrac{\rm i}{2} \, \overline{w}_+ \, (\mathcal{S}_R \mathcal{M}_-) \, \mathbf{d}_- + \tfrac{\rm i}{2} \, \overline{\mathbf{d}}_+ \, (\mathcal{S}_L \mathcal{M}_-) \, w_- - \tfrac{\rm i}{2} \, \overline{w}_+ \Big(\mathcal{S}_R \mathcal{S}_L \mathcal{M}_- \Big) w_- \Big) \; .
\end{aligned}
}
$$

Table 6: Pure spinor action: in this table we display the complete form of the pure spinor action for tyep IIA superstring in a general background. In the formulas below $\mathcal{S}_L \mathcal{M}_-$ and $mathcal S_R \mathcal{M}_-$ are given in (4.37) and $\mathcal{S}_R \mathcal{S}_L \mathcal{M}_-$ is given in (4.39) for $\chi = 0$

and it implies the following field equations on the space-time components:

$$
\begin{aligned}
0 &= \mathcal{D}_{\overline{m}} F^{\overline{m c_1 c_2 c_3}} + \frac{1}{96} \, \epsilon^{\overline{c_1 c_2 c_3 a_1 a_8}} \, F_{\overline{a_1 \dots a_4}} \, F_{\overline{a_5 \dots a_8}} \\
0 &= \Gamma^{\overline{abc}} \, \rho_{\overline{bc}} \\
R^{\overline{am}}{}_{\overline{cm}} &= 6 \, F^{\overline{a c_1 c_2 c_3}} \, F^{\overline{b c_1 c_2 c_3}} - \frac{1}{2} \, \delta^{\overline{a}}_{\overline{b}} \, F^{\overline{c_1 \dots c_4}} \, F^{\overline{c_1 \dots c_4}}
\end{aligned}
\tag{B.11}
$$

In all the above equations the overlined latin indices run on eleven values:

$$
\overline{a_1, a_2}, \dots \; = \; 0, 1, 2, \dots, 10 \tag{B.12}
$$

B.2 The type IIA FDA from circle reduction

The $D = 10$ Free Differential algebra defined in eqs. (2.1)–(2.9) and its rheonomic parametrization in eqs. (2.26)–(2.33) can now be obtained by dimensional reduction on a circle $\mathbb{S}^1$ of the algebraic structure described in the previous subsection.

Explicitly, the Kaluza-Klein ansatz realting the $D = 11$ with the $D = 10$ items is the following:

$$
\begin{aligned}
\mathbf{V}^a &= \exp\left[-\frac{1}{3} \varphi \right] V^a \\
\mathbf{V}^{11} &= \exp\left[\frac{2}{3} \, \varphi \right] \Big(d\theta \, + \, \mathbf{A}^{[1]} \Big) \\
\mathbf{A}^{[3]} &= \mathbf{C}^{[3]} \, + \, \mathbf{B}^{[2]} \, \wedge \, \Big(d\theta \, + \, \mathbf{A}^{[1]} \Big) \\
\Psi &= \exp\left[-\frac{1}{6} \varphi \right] (\psi_L \, + \, \psi_R) \, + \, (\chi_L \, + \, \chi_R) \, \exp\left[\frac{5}{6} \, \varphi \right] \Big(d\theta \, + \, \mathbf{A}^{[1]} \Big) \\
& \quad - \frac{i}{2} \exp\left[-\frac{1}{6} \varphi \right] \Gamma_r \, (\chi_L \, - \, \chi_R) \, V^r \, ,
\end{aligned}
\tag{B.13}
$$

where θ is the coordinate on the circle.

Inserting this ansatz in the $D = 11$ curvatures and redefining the $D = 10$ spin connection in such a way that the $D = 10$ torsion is zero, we obtain eqs. (2.1)–(2.9) and (2.26)–(2.33). Furthermore, from the above KK ansatz inserted in the field equations (B.11), we get the bosonic field equations of type IIA supergravity in section 2.4.

C. Conventions

In this appendix we collect all the relevant conventions for the Gamma matrix algebra utilized in the main text

$$\{\Gamma_a\,,\,\Gamma_b\}\;=\;2\,\eta_{ab}; \qquad\qquad a,b\!=\!0,\ldots,9 \tag{C.1}$$

$$\eta_{ab} = \mathrm{diag}\{+,-,-,-,-,-,-,-,-,-\} \tag{C.2}$$

$$\Gamma_0^\dagger = \Gamma_0\,, \qquad\qquad \Gamma_{11}^\dagger = \Gamma_{11}\,, \qquad \Gamma_{11}\psi_{L/R} = \pm\psi_{L/R}\,. \tag{C.3}$$

We define the charge conjiugation matrix $C\Gamma_a C^{-1} = -\Gamma_a^T$. Due to these definitions $C\Gamma_a$, $C\Gamma_{11}\,C\Gamma_{ab}$,$C\Gamma_{11bcde}$, $C\Gamma_{abcde}$ are symmetric and $C, C\Gamma_{11ab}, C\Gamma_{abc}, \Gamma_{11abc}, C\Gamma_{abcd}$ are antisymmetric.

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